Solutions Final Exam — May. 13, 2015

1. (a) (10 points) State the formal definition of a Cauchy sequence of real numbers.

A sequence, $\{a_n\}_{n\in\mathbb{N}}$, of real numbers, is Cauchy if and only if for every $\epsilon > 0$, there is a $N \in \mathbb{N}$ so that if $m, n > N$, then $|a_n - a_m| < \epsilon$.

(b) (5 points) Give an example of a sequence of real numbers, $\{a_n\}_{n\in\mathbb{N}}$, which satisfies $\lim_{n\to\infty} |a_{n+1} - a_n| \to 0$, but which is *not* Cauchy. You do not need to justify your answer.

The sequence $a_n = \sum_{i=1}^{n} \frac{1}{i}$, satisfies $|a_{n+1} - a_n| = \frac{1}{n+1}$ which goes to zero as $n \to \infty$. However, this sequence does not have a finite limit and so cannot be Cauchy (by the completeness of the reals).

(c) (15 points) Arguing directly from the definition, show that if both $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are Cauchy, then so is the sequence $\{a_n b_n\}_{n\in\mathbb{N}}$.

First observe that both $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are bounded. Indeed, by definition, there is an $N \in \mathbb{N}$ so that for all $n > N$, $|a_{N+1} - a_n| < 1$ and $|b_{N+1} - b_n| < 1$. Hence, by the triangle inequality, if $n > N$, then $|a_n| < |a_{N+1}| + 1$ and $|b_n| < |b_{N+1}| + 1$. Hence, if $M = \max\{|a_1|, |b_1|, \dots, |a_{N+1}|, |b_{N+1}|\} + 1 < \infty$ we have $|a_n| < M$ and $|b_n| < M$ for all $n \in \mathbb{N}$.

To conclude, we observe that for any $\epsilon > 0$, there is an N so that if $n, m > N$, then $|a_n - a_m| < \frac{1}{2}M^{-1}\epsilon$ and $|b_n - b_m| < \frac{1}{2}M^{-1}\epsilon$ (as both sequences are Cauchy). Hence, for any $n, m > N$

$$|a_n b_n - a_m b_m| = |a_n b_n + a_n b_m - a_n b_m + a_m b_m| \leq |a_n||b_n - b_m| + |b_m||a_n - a_m| < \epsilon.$$

This proves the claim.

2. (a) (10 points) State the formal definition of a compact subset (of $\mathbb{R}$).

> A set, A, is compact if and only if every sequence of $\{a_n\}_{n\in\mathbb{N}}$ with $a_n \in A$ possesses a finite limit point contained in A. That is, possesses a subsequence which converges to a point in A.

(b) (5 points) Give an example of a non-compact set A and a continuous function $f : A \to \mathbb{R}$ so that there is no $x_0 \in A$ so that $f(x_0) \geq f(x)$ for all $x \in A$ – i.e., f does *not* achieve its maximum. You do not need to justify your answer.

> We have shown that a set is compact if and only if it is closed and bounded. Hence, the set of integers $\mathbb{Z}$ is an example of a non-compact set. As no point of $\mathbb{Z}$ is a limit point, every function is continuous. In particular, $f(n) = n$ is continuous and unbounded from above (and so cannot achieve its maximum).

(c) (15 points) Show that if $A \subset \mathbb{R}$ is compact and non-empty and $f : A \to \mathbb{R}$ is continuous, then there is a value $x_0 \in A$ so that $f(x_0) \geq f(x)$ for all $x \in A$.

Let $B = f(A) = \{y \in \mathbb{R} : y = f(x), x \in A\}$ – this set is non-empty as A is. Let $M = \sup B \in (-\infty, \infty]$. There is a sequence, $\{b_n\}_{n\in\mathbb{N}}$ so that $b_n \in B$ and $\lim_{n\to\infty} b_n \to M$. Pick $a_n \in A$ so that $f(a_n) = b_n$. Clearly, $\{a_n\}_{n\in\mathbb{N}}$ is a sequence in A. In particular, as A is compact, there is a finite limit point $a \in A$ of this sequence. That is, there is a subsequence $\{a_{m(n)}\}_{n\in\mathbb{N}}$ so that $\lim_{n\to\infty} a_{m(n)} = a$. The continuity of f implies that

$$f(a) = f(\lim_{n\to\infty} a_{m(n)}) = \lim_{n\to\infty} f(a_{m(n)}) = \lim_{n\to\infty} b_n = M.$$

This proves the claim with $x_0 = a$.

3. (a) (10 points) State the mean value theorem.

For $a < b$ and a continuous function $f : [a, b] \to \mathbb{R}$ which is differentiable at each point of (a, b), there is a value $c \in (a, b)$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(b) (5 points) Give an example of a function $f : (-1, 1) \to \mathbb{R}$ with the property that there is no differentiable function $F : (-1, 1) \to \mathbb{R}$ with $F' = f$. You do not need to justify your answer.

The function

$$f(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

cannot be the derivative of any function as the derivative of a differentiable function must satisfy the conclusions of the intermediate value theorem.

(c) (15 points) Show that if $f : (a, b) \to \mathbb{R}$ is differentiable and $\sup_{x\in(a,b)} |f'(x)| < C$, then for all $x, y \in (a, b)$, $|f(x) - f(y)| \leq C|x - y|$.

If $x = y$, then this is immediate. If $x \neq y$, then this follows immediately from the mean value theorem applied to f on the interval $[x, y]$ (when $x < y$ – if $y < x$ apply it on $[y, x]$).

4. (a) (10 points) State one of the (equivalent) definitions of a function $f : [a, b] \to \mathbb{R}$ being Riemann integrable.

f is Riemann integrable if it is bounded and for every $\epsilon > 0$, there is a $\delta > 0$, so that if P is a partition with $|P| < \delta$, then $Osc(f, P) = S^+(f, P) - S^-(f, P) < \epsilon$.

(b) (10 points) Give an example of a function $f : [0, 1] \to \mathbb{R}$ which is *not* Riemann integrable. You do not need to justify your answer.

Consider Dirichlet's function $f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$ This is not Riemann integrable on $[0, 1]$ as the upper sum (for any partition) is always 1 while the lower sum is always 0. That is, the osciallation is always 1 no matter the partition.

(c) (20 points) Using the definition from (a) directly, show that if $f : [a, b] \to \mathbb{R}$ is continuous, then it is Riemann integrable.

As f is continuous and $[a, b]$ is compact, f is uniformly continuous and is bounded. Using the uniform continuity of f, given an $\epsilon > 0$, pick $\delta > 0$ so that $|x - y| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{b-a}$. For any partition, $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ we have $S^+(f, P) = \sum_{i=1}^n M_i(x_i - x_{i-1})$ where $M_i = \sup_{[x_{i-1},x_i]} f(x)$ and $S^-(f, P) = \sum_{i=1}^n m_i(x_i - x_{i-1})$ where $m_i = \inf_{[x_{i-1},x_i]} f(x)$. By the continuity of f and compactness of $[x_{i-1}, x_i]$ we have $M_i = f(a_i)$ and $m_i = f(b_i)$. Hence, if $|P| < \delta$, then $M_i - m_i < \frac{\epsilon}{b-a}$ as $|a_i - b_i| < \delta$ and so

$$Osc(f, P) = \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) < \sum_{i=1}^{n} \frac{\epsilon}{b - a}(x_i - x_{i-1}) = \epsilon.$$

Where the first inequality used that $x_i - x_{i-1} \geq 0$.

5. (a) (15 points) State both directions of the fundamental theorem of calculus

Integration is the inverse of differentiation: If $f : I \to \mathbb{R}$ is C^1 and $x_0 \in I$ for some interval I, then for all $x \in I$, $f(x) - f(x_0) = \int_{x_0}^{x} f'(t)dt$
Differentiation is the inverse of integration. If $f : I \to \mathbb{R}$ is continuous for some interval I and $F(x) = \int_{x_0}^{x} f(t)dt$ for some $x_0 \in I$. Then F is C^1 and $F'(x) = f(x)$.

(b) (5 points) Give a Riemann integrable function, $f : [-1,1] \to \mathbb{R}$, for which the function $F(x) = \int_0^x f(t)dt$ is not differentiable at some point of $(-1,1)$. You do not need to justify your answer.

The Heaviside function

$$f(x) = \begin{cases} 1 & x < 0 \\ 0 & x \geq 0 \end{cases}$$

has a single jump discontinuity and so is Riemann integrable. For this function $F(x) = \frac{1}{2}(|x| + x)$ and this function is not differentiable at $x = 0$.

(c) (10 points) Suppose $f, g : (a,b) \to \mathbb{R}$ are C^1 and that $[c,d] \subset (a,b)$. Show that

$$\int_c^d f'(x)g(x)dx = f(d)g(d) - f(c)g(c) - \int_c^d f(x)g'(x)dx.$$

By the Leibniz rule $h(x) = f(x)g(x)$ is differentiable and its derivative is $h'(x) = f'(x)g(x) + f(x)g'(x)$. This is a continuous function – that is, h is C^1 – indeed, both f' and g are continuous and so their product is, the same is true of f and g' and so h' is the sum of two continuous functions. Hence, we may apply the fundamental theorem of calculus to h' and so obtain

$$h(d) - h(c) = \int_c^d h'(x)dx = \int_c^d f'(x)g(x) + f(x)g'(x)dx$$

That is,

$$f(d)g(d) - f(c)g(c) = \int_c^d f'(x)g(x) + f(x)g'(x)dx$$

and we obtain the result by rewriting things.

6. (a) (10 points) Fix an interval $I \subset \mathbb{R}$ and let $f_n : I \to \mathbb{R}$, $n \in \mathbb{N}$, and $f : I \to \mathbb{R}$ be functions. State the definition of f_n converging uniformly to f.

The functions f_n converge uniformly to f if and only if

$$\lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| \to 0.$$

(b) (10 points) Give an example of a power series $\sum_{n=0}^{\infty} a_n x^n$ which converges pointwise on $(-1, 1)$ but not uniformly. You do not need to justify your answer.

The geometric series

$$\sum_{n=0}^{\infty} x^n$$

can be check to converge at each point $x \in (-1, 1)$ to the value $\frac{1}{1-x}$. This convergence cannot be uniform as the uniform limit of uniformly continuous functions must be uniformly continuous. Clearly, each partial sum is uniformly continuous on $(-1, 1)$ (as they are polynomials), however the function $1/(1 - x)$ is not uniformly continuous.

(c) (20 points) Fix an interval $I \subset \mathbb{R}$ and let $f_n : I \to \mathbb{R}$, $n \in \mathbb{N}$, be functions which satisfy

1. for all $x \in I$ and $n \in \mathbb{N}$, $0 \le f_{n+1}(x) \le f_n(x)$, and
2. for all $x \in I$, $\lim_{n\to\infty} \sup_{x\in I} f_n(x) = 0$.

Show that the series $\sum_{n=1}^{\infty}(-1)^n f_n(x)$ converges uniformly on I. Hint: show that for $m > N$:

$$0 \le (-1)^N \sum_{k=N}^{m} (-1)^k f_k(x) \le f_N(x).$$

We first observe that for each fixed $x_0 \in I$, the series

$$\sum_{n=1}^{\infty}(-1)^n f_n(x_0)$$

converges. To see this note that the two conditions 1) and 2) imply that this series satisfies the alternating series test. Indeed, $\lim_{n\to\infty} f_n(x_0) = 0$ – this is because 1) implies $\liminf_{n\to\infty} f_n(x_0) \ge 0$ and 2) implies $\limsup_{n\to\infty} f_n(x_0) \le 0$. To see this directly fix $x \in I$ and set

$$S_N(x_0) = \sum_{n=1}^{N}(-1)^n f_n(x_0)$$

and observe that $S_{2N+2}(x_0) \le S_{2N}(x_0)$ and $S_{2N+3}(x_0) \ge S_{2N+1}(x_0)$ and $S_{2N+1}(x_0) \le S_{2N}(x_0)$. Hence, the sequence $\{S_{2N}(x_0)\}_{N\in\mathbb{N}}$ is monotone non-increasing and is bounded from below and so converges to some finite limit $S_+(x_0)$. Likewise, $\{S_{2N+1}(x_0)\}_{N\in\mathbb{N}}$ is bounded from above and non-decreasing and so converges to some finite limit $S_-(x_0)$. As $S_{2N+1} - S_{2N} = (-1)^{2N+1} f_{2N+1}(x_0)$ is tending to zero, we have that $S_+(x_0) = S_-(x_0)$ and that the partial sums converge to this common value.

Hence, there is a well defined function $f : I \to \mathbb{R}$ be given by $f(x) = \sum_{n=1}^{\infty}(-1)^n f_n(x)$ and the S_n converge pointwise to f. To see the uniform convergence, observe now that for a fixed N and for all $m > N$ and $x_0 \in I$

$$(-1)^N \sum_{k=N}^{m}(-1)^k f_k(x_0) \le f_N(x_0).$$

Indeed, this follows from 1) and an induction argument (it is immediate when $m = N+1$ and $m = N+2$ – the induction is straightforward). Hence,

$$0 \le (-1)^N \sum_{k=N}^{m}(-1)^k f_k(x_0) \le \sup_{x\in I} f_N(x)$$

In other words, $f(x_0) - S_{2N-1}(x_0) \le \sup_{x\in I} f_{2N}(x)$ and $f(x_0) - S_{2N}(x_0) \ge -\sup_{x\in I} f_{2N+1}(x)$. That is,

$$\sup_{x\in I} |f(x) - S_n(x)| \le \sup_{x\in I} f_{n+1}(x)$$

since the right hand side tends to zero by 2) so does the left hand side which proves the claim.