Math 405 Midterm Exam 1 — Feb. 23, 2015

Name: ____

- Complete the following problems. In order to receive full credit, please make sure to *justify your answers*. You are free to use results from class or the course textbook as long as you clearly state what you are citing.
- **You have 75 minutes.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted (nor are they needed). If you finish early, you must hand your exam paper to a proctor.
- Please check that your copy of this exam contains 4 numbered pages and is correctly stapled.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

The following boxes are strictly for grading purposes. Please do not mark.

Question	Points	Score
1	25	
2	25	
3	25	
4	25	
Total	100	

1. Let $\sim$ be an equivalence relation on $\mathbb{N}$.

   (a) (10 points) Give the formal definition the equivalence class $[n]$ of an element $n \in \mathbb{N}$ with respect to $\sim$ and the formal definition of the quotient space $\mathbb{N}/\sim$ as a subset of the power-set $P(\mathbb{N})$.

   (b) (15 points) Show that if each equivalence class of $\sim$ contains only finitely many elements, then the quotient space $X = \mathbb{N}/\sim$ is countable.

2. (a) (10 points) State the formal definition of Cauchy sequence (of real numbers).

   (b) (15 points) Show directly from the definition, that if $\{a_n\}$ is a Cauchy sequence, then $\{a_n\}$ is bounded in the sense that there is an $N > 0$ so that $|a_n| < N$ for all $n \in \mathbb{N}$.

3. (a) (10 points) Give an example of a set $X \subset \mathbb{R}$ which is bounded from above and does not contain its least upper bound.

   (b) (15 points) Let $A, B \subset \mathbb{R}$ be non-empty sets which are both bounded from above. Define $A+B = \{z \in \mathbb{R} : z = a + b \text{ for some } a \in A \text{ and some } b \in B\}$. Show that $\sup(A + B) = \sup A + \sup B$.

4. (a) (10 points) Give an example of bounded sequences $\{x_n\}$ and $\{y_n\}$ so that x is a limit point of $\{x_n\}$ and y is a limit point of $\{y_n\}$, but $x + y$ is *not* a limit point of the sequence $\{x_n + y_n\}$.

   (b) (15 points) Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded sequences. Show that if z is a limit point of $\{x_n + y_n\}$, then there is a limit point x of $\{x_n\}$ and a limit point y of $\{y_n\}$ so that $z = x + y$. (Hint: Use the fact that limit points are limits of subsequences and that subsequences of convergent sequences converge).

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