Solutions Midterm Exam 1 — Mar. 5, 2014

1. Determine whether the following statements are true or false. Justify your answer (i.e., prove the claim, derive a contradiction or give a counter-example).

   (a) (10 points) If $A \subset B$, and B is countable, then A is countable.

   > False. A may be finite.

   (b) (10 points) If $\mathcal{B}$ is an open cover of $(0,1]$, then $\mathcal{B}$ has a finite subcover.

   > False. The cover $\mathcal{B} = \{(2/(n+2), 2/n) : n \in \mathbb{N}\}$ cannot have a finite subcover. Indeed, if $\mathcal{B}'$ was a finite subcover, then, there would be an N so that if $I \in \mathcal{B}'$, then $I = (2/(n+2), 2/n)$ for some $n < N$. This would mean that the value $2/(N+3) \in (0,1]$ was not in any element of $\mathcal{B}'$ – that is, $\mathcal{B}'$ could not itself be a cover of $(0,1]$.

(c) (10 points) If $[0,1] \supset I_1 \supset I_2 \supset \ldots \supset I_n \supset \ldots$ is a nested sequence of closed intervals, then $\cap_{n=1}^{\infty} I_n$ is non-empty.

True. As each $I_i \subset [0,1]$ they are all bounded. Hence, the I_k are all closed and bounded intervals and so compact. By definition a closed interval is of the form $I = [a,b]$ for $a \leq b$ and so is non-empty. Hence, their intersection is non-empty.

(d) (10 points) For non-empty $A, B \subset \mathbb{R}$, let $A + B = \{x + y : x \in A, y \in B\}$. If A is open, then $A + B$ is open.

True. Pick $z \in A+B$ and write $z = x+y$. As A is open, there is an ϵ so that $(x-\epsilon, x+\epsilon) \subset A$. Hence, $(z-\epsilon, z+\epsilon) = (x-\epsilon, x+\epsilon) + \{y\} \subset A+B$. That is, $A+B$ contains a neighborhood of each of its points and so is open.

(e) (10 points) Given sequences $\{x_n\}$ and $\{y_n\}$, define a new sequence $\{z_n\}$ by $z_{2n} = x_n$ and $z_{2n-1} = y_n$. The sequence $\{z_n\}$ converges if and only if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ – that is, both sequences converge and have the same limit.

True. If $\{z_n\}$ converges, then all subsequences – such as, $\{x_n\}$ and $\{z_n\}$ – converge to the same limit. Conversely, if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x$, then for any $\epsilon > 0$, there is an m so that $m < n$, implies that $|x_n - x| < \epsilon$ and $|y_n - x| < \epsilon$. Hence, if $2(m+1) < n$, $|z_n - x| < \epsilon$. That is, $\lim_{n\to\infty} z_n = x$.

(f) (10 points) If $f : D \to \mathbb{R}$ is a continuous function with domain $D \subset \mathbb{R}$, then for all $x_0 \in \bar{D}$, the closure of D, $\lim_{x\to x_0} f(x)$ exists.

False. Consider $D = (-1, 0) \cup (0, 1)$ and $f(x) = 1/x$, then f is continuous but $\lim_{x\to 0} f(x)$ does not exist.

2. (20 points) Let $\{a_n\}$ be a Cauchy sequence, with $a_n \geq a > 0$. Working directly from the definitions, show that $\left\{a_n^{-2}\right\}$ is Cauchy.

We note that $\left|\frac{1}{a_n^2} - \frac{1}{a_k^2}\right| = \frac{|a_n - a_k||a_n + a_k|}{|a_n^2 a_k^2|} \leq 2Na^{-4}|a_n - a_k|$ where $N > 0$ is some number so that $|a_n|, |a_k| \leq N$ and we used that $a_n, a_k \geq a > 0$. As Cauchy sequences are bounded, there is an N so that, for all n, $|a_n| \leq N$. Now, given, any $\epsilon > 0$, as $\{a_n\}$ is Cauchy, there is an m, so that if $m < n, k$, then $|a_n - a_k| < \frac{1}{2}N^{-1}a^4\epsilon$. Hence, $\left|\frac{1}{a_n^2} - \frac{1}{a_k^2}\right| \leq 2Na^{-4}\left(\frac{1}{2}N^{-1}a^4\epsilon\right) = \epsilon$. That is, $\left\{a_n^{-2}\right\}$ is Cauchy.

3. (a) (5 points) Let $S = \{x \in \mathbb{R} : x^3 < x\}$. Determine $\sup S$ and $\inf S$.

We note that if $x > 0$, then $x \in S$ if and only if $x^2 < 1$, that is $x \in (0, 1)$. Likewise, if $x < 0$, then $x \in S$ if only if $x^2 > 1$, that is $x \in (-\infty, -1)$. Clearly, $0 \notin S$, so $S = (-\infty, -1) \cup (0, 1)$. Hence, $\sup S = 1$ and $\inf S = -\infty$.

(b) (15 points) Let $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$, $a_1 = 1$. Set $A = \{x \in \mathbb{R} : x = a_n, n \in \mathbb{N}\} \subset \mathbb{R}$. Determine, $\limsup_{n\to\infty} a_n, \liminf_{n\to\infty} a_n, \inf A, \sup A$ and all limit points (if any) of A. (Hint: Show that, for $n \geq 1$, $2 \leq a_{n+1}^2$.)

We note that for $n \geq 1$, $a_{n+1}^2 \geq 2$ and that for $n \geq 2$, $a_{n+1} \leq a_n$. To see the former we note that $a_{n+1}^2 = \frac{1}{4}(a_n + \frac{2}{a_n})^2 = \frac{1}{4}(a_n - \frac{2}{a_n})^2 + 2 \geq 2$. The latter then follows from $\frac{2}{a_n} \geq a_n$ for $n \geq 2$. From this we conclude that $a_1 = 1$ and $a_2 = \frac{3}{2}$ are, respectively, upper and lower bounds for A and so $\sup A = \frac{3}{2}$ and $\inf A = 1$. For $n \geq 2$, a_n is a bounded monotone non-increasing sequence and hence $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n = a$ for some $a \in \mathbb{R}$. One verifies, that $a^2 = 2$ and that $a_n \geq 0$ and so conclude that $a = \sqrt{2}$. The only possible limit point of A is $\sqrt{2}$, this is indeed a limit point as each a_n is necessarily rational and $\sqrt{2}$ is irrational and so there are points in A different from $\sqrt{2}$ arbitrarily close to $\sqrt{2}$ but different from it.