Mathematic 405, Fall 2019: Assignment #3

Due: **Wednesday, September 25th**

Instructions: Please ensure that your answers are legible. Also make sure that sufficient steps are shown. Page numbers refer to the course text.

Problem #1.

a) Show that any closed interval $I = [a, b]$ is the countable infinite intersection of open intervals.
b) Show that any open interval $I = (a, b)$ is the countable infinite union of closed intervals.

Problem #2. Define an *open subset of* $\mathbb{R}$, $U \subset \mathbb{R}$, to be set that has the property that for every $x \in U$, there is an open interval I so that $x \in I \subset U$. The empty set is considered open as it vacuously satisfies this condition.

a) Show that if $\{U_\lambda\}_{\lambda \in A}$ is any collection of open subsets, then $U = \bigcup_{\lambda \in A} U_\lambda$ is open.
b) Show that if $U_1, \ldots, U_n$ is a finite collection of open subsets, then $U = U_1 \cap \ldots \cap U_n$ is open.
c) Show by example, that there is a countable infinite collection of open subsets $\{U_n\}_{n \in \mathbb{N}}$ so that $V = \bigcap_{i=1}^{\infty} U_i$ is not open.

Problem #3. Let U be an open subset as defined above. Show that if U is non-empty, then U contains some rational number.

Problem #4. Determine if the following series converge and if they do find their limit. Please justify your answer rigorously.

a) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$.
b) $\left\{\frac{(-2)^n}{n^2}\right\}_{n=1}^{\infty}$.

Problem #5. Let $\{x_n\}_{n=1}^{\infty}$ be a convergent monotone sequence. If $\lim_{n \to \infty} x_n = x_k$ for a fixed k, then $x_n = x_k$ for all $n \geq k$.

Problem #6.

a) Let $\{I_n\}_{n \in \mathbb{N}}$ be a collection of closed bounded intervals with $I_{n+1} \subset I_n$ for all $n \geq 1$. Show that $\bigcap_{n=1}^{\infty} I_n$ is non-empty.
b) Let $\{J_n\}_{n \in \mathbb{N}}$ be a collection of open bounded intervals with $J_{n+1} \subset J_n$ for all $n \geq 1$. Show by example that it is possible for $\bigcap_{n=1}^{\infty} J_n$ to be empty.