Mathematic 405, Fall 2019: Assignment #6

Due: **Wednesday, October 23th**

Instructions: Please ensure that your answers are legible. Also make sure that sufficient steps are shown. Page numbers refer to the course text.

Problem #1. For $S \subset \mathbb{R}$ let $f, g : S \to \mathbb{R}$ be continuous functions. Show that $M(x) = \max\{f(x), g(x)\}$ and $m(x) = \min\{f(x), g(x)\}$ are both continuous functions.

Problem #2. Let $f : [0,1] \to [0,1]$ be a continuous function. Show that there is a value $c \in [0,1]$ so that $f(c) = c$. Such a c is called a *fixed point* of f.

Problem #3. Suppose $f : [0,1] \to (0,1)$ is a continuous.

a) Given an example of such an f.
b) Show that no such f is onto.

Problem #4. Set $f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

a) Show that f is discontinuous.
b) Show that f has the intermediate value property. That is, for any choice of $a < b$, if $f(a) < y < f(b)$, then there is a $c \in (a,b)$ so $f(c) = y$.

Problem #5. Recall, the characteristic polynomial of a $n \times n$ matrix, A, is given by $p_A(t) = \det(tI_n - A)$ where I_n is the $n \times n$ identity matrix

a) Show that if n is odd that $p_A(t)$ has at least one real root (and so A has at least one eigenvector).
b) Show that if n is even and $\det(-A) = \det(A) < 0$, then p_A has at least two real roots.

Problem #6.

a) Show by example that if $f : (0,1) \to \mathbb{R}$ is continuous and $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence with $x_n \in (0,1)$, then $\{f(x_n)\}_{n=1}^{\infty}$ need not be Cauchy.
b) Show that if $g : \mathbb{R} \to \mathbb{R}$ is continuous and $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then so is $\{f(x_n)\}_{n=1}^{\infty}$.

Problem #7. Let $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. Show that f is differentiable at $x = 0$, but discontinuous everywhere else.

Problem #8. Let $f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

a) Show that f is differentiable at every $x \in \mathbb{R}$ and determine its derivative.
b) Show that f' is not continuous at $x = 0$.