Mathematic 405, Fall 2019: Assignment #7

Due: **Wednesday, October 30th**

Instructions: Please ensure that your answers are legible. Also make sure that sufficient steps are shown. Page numbers refer to the course text.

Problem #1. Let I be a open interval and $x_0 \in I$. Suppose $f : I \to \mathbb{R}$ is differentiable at $x_0 \in I$ and let $g(x) = f(x_0) + m(x - x_0)$ be an affine function

a) Show that there is a $\delta > 0$ and a $C > 0$ so that if $x \in I$ has $|x - x_0| < \delta$, then $|f(x) - g(x)| \leq C|x - x_0|$.
b) Show that $m = f'(x_0)$ if and only if for every $\epsilon > 0$, there is a $\delta > 0$ so that $x \in I$ and $|x - x_0| < \delta$ implies $|f(x) - g(x)| \leq \epsilon|x - x_0|$.

Problem #2. Show that there is no differentiable function $f : (-2, 2) \to \mathbb{R}$ so that

$$f'(x) = \begin{cases} 1 & x \in (-2, 0) \\ -1 & x \in [0, 2) \end{cases}$$

Problem #3. Suppose there is an $\epsilon > 0$ and a $C > 0$ so that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq C|x - y|^{1+\epsilon}$ for every $x, y \in \mathbb{R}$. Show that f is constant.

Problem #4. Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous and differentiable on $(0, 1)$. Moreover, suppose that $f(0) = 0$ and $f'(x) \leq x$ for $x \in (0, 1)$.

a) Use the Mean Value Theorem to show that $f(x) \leq x^2$.
b) Show that in fact $f(x) \leq \frac{x^2}{2}$. Hint: consider the function $g(x) = f(x) - \frac{x^2}{2}$.

Problem #5. We say a function $f : [a, b] \to \mathbb{R}$ is *convex* if, for all $a \leq x < y \leq b$ and $t \in [0, 1]$, $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$.

a) Show that if f is convex on $[a, b]$ and has a relative minimum at $c \in (a, b)$, then c is an absolute minimum.
b) Show that if f is convex on $[a, b]$ and has an absolute maximum at $c \in (a, b)$, then f is constant.

Problem #6. Show that if $f : [a, b] \to \mathbb{R}$ is continuous and twice differentiable on (a, b) and $f''(x) \geq 0$ for all $x \in (a, b)$, then f is convex as defined above. Hint: Fix x, y and consider the function $g(t) = f(tx + (1 - t)y) - tf(x) + (1 - t)f(y)$.

Problem #7. Let $f(x) = |x|^3$ on $\mathbb{R}$. Compute $f'(x)$ and $f''(x)$, but show $f'''(0)$ does not exist.

Problem #8. Do Exercise 5.1.1 of Lebl's book.