Mathematic 405, Fall 2019: Assignment #8

Due: **Wednesday, November 6th**

Instructions: Please ensure that your answers are legible. Also make sure that sufficient steps are shown. Page numbers refer to the course text.

Problem #1. Let $f : [a.b] \to \mathbb{R}$ be Riemann integrable. Show that for any $\epsilon > 0$, there is a partition $P = \{x_0, x_1, \dots, x_n\}$ so that for any choice of sample points $x_k^* \in [x_{k-1}, x_k]$, $1 \le k \le n$, one has

$$\left| \int_a^b f(x)dx - \sum_{k=1}^{n} f(x_k^*)\Delta x_k \right| < \epsilon.$$

This sum is called a *Riemann sum.*

Problem #2. Show the mean value theorem for integrals. That is show that if $f : [a, b] \to \mathbb{R}$ is continuous, then there is a $c \in [a, b]$ so that

$$\int_a^b f(x)dx = f(c)(b - a).$$

Problem #3. Show that if $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ is equal to f except possibly at $c \in [a, b]$ (i.e. $f(x) = g(x)$ for $x \in [a, b], x \neq c$), then g is Riemann integrable and

$$\int_a^b f(x)dx = \int_a^b g(x)dx.$$

Problem #4. Show that if $f : [a, b] \to \mathbb{R}$ is continuous, then the triangle inequality holds

$$\left| \int_a^b f(x)dx \right| \le \int_a^b |f(x)|dx.$$

Why is this problem much harder if you only assume f is Riemann integrable? (You do not need to prove this case).

Problem #5. Show that if $f : [a, b] \to \mathbb{R}$ is monotone increasing, then f is Riemann integrable.

Problem #6. Let $f : [a, b] \to \mathbb{R}$ be continuous.

a) Show that if $\int_a^b (f(x))^2 dx = 0$, then f is identically zero.
b) Show that if $\int_a^b f(x)\phi(x)dx = 0$ for every $\phi \in C^0([a, b])$ with $\phi(a) = \phi(b) = 0$, then f is identically zero.

Problem #7. Prove the integration by parts formula, that is show that if F and G are C^1 functions, on (c, d) and $[a, b] \subset (c, d)$, then

$$\int_a^b F(x)G'(x)dx = F(b)G(b) - F(a)G(b) - \int_a^b F'(x)G(x)dx.$$

Problem #8. Use integration to show that if $f : (-2, 2) \to \mathbb{R}$ is C^1 and $f(0) = 0$ and $f'(x) \ge x$ for all $x \in (-2, 2)$, then $f(x) \ge \frac{1}{2}x^2$ on $[0, 2)$. What happens on $(-2, 0]$?