Mathematic 405, Fall 2019: Assignment #9

Due: **Wednesday, November 20th**

Instructions: Please ensure that your answers are legible. Also make sure that sufficient steps are shown. Page numbers refer to the course text.

Problem #1. Let $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ be bounded functions. Show that

a) $\|f + g\|_u \leq \|f\|_u + \|g\|_u$.
b) $\|fg\|_u \leq \|f\|_u \|g\|_g$.

Problem #2. Let $f_n, g_n : S \to \mathbb{R}$ be two sequences of bounded functions and let $h_n = f_n + g_n$. Show the following.

a) If $f_n \to f$ and $g_n \to g$ pointwise, then $h_n \to f + g$ pointwise
b) If $f_n \to f$ uniformly and $g_n \to g$ uniformly, then $h_n \to f + g$ uniformly.

Problem #3. Let $f_n : (a, b) \to \mathbb{R}$ be a sequence of non-decreasing functions (so $x < y$ implies $f_n(x) \leq f_n(y)$). Show that if $f_n \to f$ pointwise, then f is also non-decreasing.

Problem #4. Show that if $f : (-\infty, \infty) \to \mathbb{R}$ is uniformly continuous and $f_n(x) = f(x + \frac{1}{n})$, then $f_n \to f$ uniformly on $(-\infty, \infty)$.

Problem #5. Let $h \in C^0([a, b])$ and set

$$f(x) = \begin{cases} h(a) & x < a \\ h(x) & x \in [a, b] \\ h(b) & x > b. \end{cases}$$

Using this f set,

$$f_n(x) = \frac{n}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(t)dt$$

a) Show that f is uniformly continuous and bounded.
b) Show that $f_n \in C^1((-\infty, \infty))$ is bounded and satisfies $\|f_n\|_u \leq \|f\|_u$.
c) Show that $f_n \to h$ uniformly on $[a, b]$.

Problem #6. Let

$$\phi(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

a) Show that for any polynomial P, the following function is continuous,

$$f(x) = P(1/x)\phi(x) = \begin{cases} P(1/x)e^{-1/x} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

b) Use a) and mathematical induction to show that, for all $k \geq 1$, the k-th derivative of ϕ satisfies $\phi^{(k)}(x) = P_k(1/x)\phi(x)$ where P_k is some polynomial. Conclude that $\phi \in C^\infty((-\infty, \infty))$.

Problem #7. Let ϕ be the function from the preceding exercise.

a) Using ϕ show that for every $a < b$, there is a function $\psi \in C^\infty((-\infty, \infty))$ so that $\psi(x) > 0$ on (a, b) and $\psi(x) = 0$ for $x \notin (a, b)$.
b) Using ψ show that for any $a < b$ there is a function $\eta \in C^\infty((-\infty, \infty))$ so that $0 \leq \eta \leq 1$ and $\eta(x) = 0$ for $x \leq a$ and $\phi(x) = 1$ for $x \geq b$. (Hint: What happens when you integrate ψ?).
c) Using η show that for $a < c < d < b$ there is a function $\zeta \in C^\infty((-\infty, \infty))$ that satisfies $0 \leq \zeta \leq 1$ and $\zeta(x) = 1$ for $c \leq x \leq d$ and $\zeta(x) = 0$ for $x \leq a$ and $x \geq b$.