1. Let $y_n = \lim_{n\to\infty} x_{n,m}$ and we shall show that $\lim_{n\to\infty} y_n = L$. The other case is similar. The sequence $\{y_n\}$ is well-defined by assumption. Let $\varepsilon > 0$, by definition of the joint limit there is $M_1 \in \mathbb{N}$ such that $m, n \geq M_1 \implies |x_{n,m} - L| < \varepsilon/2$. By definition, for a fixed n, there is $M_2 \in \mathbb{N}$ such that $m \geq M_2 \implies |x_{n,m} - y_n| < \varepsilon/2$. Thus by the triangle inequality, for $n \geq M_1$,

$$|y_n - L| \leq |y_n - x_{n,m}| + |x_{n,m} - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

where m is chosen so that $m \geq \max\{M_1, M_2\}$, so $\lim_{n\to\infty} y_n = L$.

2. (a) Let $x, y \in I$ and let $\varepsilon > 0$, and suppose that $K_n \leq K$ for all n. Since $f_n \to f$ uniformly, there is $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(z) - f(z)| \leq |x - y|$ for all $z \in I$. Thus by the triangle inequality

$$\begin{aligned}|f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq |x - y| + K\,|x - y| + |x - y| = (K + 2)\,|x - y|\end{aligned}$$

where n is chosen so that $n \geq N$.

(b) Consider $x_n = 4^{-n}$ and $y_n = 4^{-n-1}$ for $n \in \mathbb{N}$, then

$$|\sqrt{x_n} - \sqrt{y_n}| = \left|2^{-n} - 2^{-n-1}\right| = \frac{1}{2^{n+1}}$$

But on the other hand

$$|x_n - y_n| = \left|4^{-n} - 4^{-n-1}\right| = \frac{3}{4^{n+1}}$$

So for this particular choice we have

$$|\sqrt{x_n} - \sqrt{y_n}| \leq C\,|x_n - y_n| \implies 2^{n+1} \leq 3C \implies C \geq \frac{2^{n+1}}{3}$$

Since this C goes to infinity as $n \to \infty$, there is no universal constant C satsifying the condition, so f is not Lipschitz.

(c) Consider

$$f_n(x) = \begin{cases} \sqrt{x} & 1 \geq x \geq 4^{-n} \\ 2^{-n} & 4^{-n} > x \geq 0 \end{cases}$$

Clearly f_n is continuous. We check that f_n is Lipschitz. Let $x, y \in [0, 1]$. We have that

$$|f_n(x) - f_n(y)| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \max_{x,y}\left\{\frac{1}{\sqrt{x} + \sqrt{y}}\right\}|x - y| = 2^{n-1}\,|x - y|$$

so $f_n(x)$ is Lipschitz with Lipschitz constant 2^{n-1}. Finally we check that $f_n \to f$ uniformly. Let $\varepsilon > 0$. Choose N such that $2^{-N} < \varepsilon$. Then for $n \geq N$ we have

$$|f_n(x) - f(x)| \leq \begin{cases} 0 & 1 \geq x \geq 4^{-n} \\ 2^{-n} & 4^{-n} > x \geq 0 \end{cases} < \varepsilon$$

which proves the uniform convergence.

3. Define $S(x) = \sum_{n=1}^{\infty} (-1)^n f_n(x)$. By alternating series test the function S is well-defined (the pointwise limit always exists). It remains to show that $S_N \to S$ uniformly. Let $\varepsilon > 0$. Since $f_n \to f$ uniformly there is $N_0 \in \mathbb{N}$ such that $n \geq N_0$ implies $f_n < \varepsilon$. Thus for $N \geq N_0$ (due to the monotonicity of the even-numbered and odd-numbered terms)

$$|S_N(x) - S(x)| \leq |f_{N+1}(x)| < \varepsilon$$

which means $S_N \to S$ uniformly. The continuity of S follows from uniform convergence.

4. Let $f_n(x) = \frac{x^n}{n}$ for $x \in [0,1]$. It is clear that $f_{n+1}(x) \leq f_n(x)$ (numerator is decreasing and denominator is increasing), and that $f_n(x) \to 0$ uniformly. Thus by the previous problem the alternating series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$$

converges uniformly on $[0,1]$, which is equivalent to $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converging uniformly on $[-1,0]$.
To evaluate the series at $x = -1$, we calculate for $x \in (-1,0)$

$$S(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \implies S'(x) = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Note here term by term differentiation is justified by uniform convergence (essentially interchanging limits). Thus

$$S(x) = \int \frac{1}{1-x} dx = -\log(1-x)$$

for $x \in (-1,0)$. Note that by the previous problem S is continuous on $[-1,0]$. Since $S(x) = -\log(1-x)$ on $(-1,0)$ we conclude

$$S(-1) = \lim_{x \to -1} -\log(1-x) = -\log 2$$

5. For $x < -1$ we compute

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \implies a_n = \frac{1}{(1-x)^{n+1}}$$

where a_n is the coefficient of the power series representation of f centered at x. Thus the radius of convergence at x is

$$r = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}} = 1 - x$$

which exists and is strictly positive for $x < 1$. So f is real analytic on $(-\infty, 1)$.

6. (a) f is C^1 as $f(c) = f'(c) = 0$. This function clearly satisfies the IVP when $x < c$. For $x \geq c$ we have

$$y'(x) = \frac{1}{2}(x-c) = \sqrt{y(x)}$$

so the IVP is satisfied on all of $\mathbb{R}$.

(b) $y = 0$.

(c) This is not a contradiction because the right hand side of the equation is not Lipschitz continuous (as seen in Problem 2).