1. (a) $[a,b] = \bigcap_{n=1}^{\infty}(a-1/n, b+1/n)$.

   (b) Let N be large enough so that $b-a > \frac{2}{N}$ (so that we do not get empty set in the union, but it is also fine if you include many emtpy sets). Then $(a,b) = \bigcup_{n=N}^{\infty}[a+1/n, b-1/n]$.

2. (a) Let $x \in U$, then by definition there is some λ such that $x \in U_\lambda$. Since U_λ is open we have an open interval I_x such that $x \in I_x \subset U_\lambda \subset U$, so U is open.

   (b) Let $x \in U$, then $x \in U_i$ for each $i = 1, \ldots, n$. By definition there are open intervals $I_1, \ldots I_n$ such that $x \in I_i \subset U_i$. Without loss of generality, and after possibly shrinking I_i, we can assume $I_i = (x-a_i, x+a_i)$ for $a_i \in \mathbb{R}$. Now choose a real number $a < \min\{a_1, \ldots, a_n\}$, it follows that $(x-a, x+a) \subset I_i \subset U_i$ for each $i = 1, \ldots, n$, and hence $(x-a, x+a) \subset \bigcap_{i=1}^{n} U_i = U$, so U is open.

   (c) The example is the same as Problem 1 (a). We will show that the closed interval (as the name suggests) $[a,b]$ is not open. In fact, any open interval around b will not be fully contained in $[a,b]$. Indeed, any interval around b contains an interval of the form $(b-\varepsilon, b+\varepsilon)$ for some $\varepsilon > 0$, and it is clear that $b + \varepsilon/2 \notin [a,b]$.

3. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, given any $x \in U$ and an interval $(x-a, x+a) \subset U$, $\mathbb{Q} \cap (x-a, x+a)$ is not empty, so there is at least some rational number in it.

4. (a) We claim that the limit is 1. Indeed, given $\varepsilon > 0$, let $N > 0$ be such that $\frac{1}{N} < \varepsilon$, then for all $n > N$ we have

$$\left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} < \frac{1}{N} < \varepsilon$$

   So the limit is 1 by definition.

   (b) We claim that the limit does not exist. Suppose for a contradiction that $L \in \mathbb{R}$ is the limit of the sequence. We must exhibit an ε such that there is no $N \in \mathbb{N}$ such that

$$\left|\frac{(-2)^n}{n^2} - L\right| < \varepsilon$$

   for all $n > N$. We will show this using $\varepsilon = 1$. Let n be even and by triangle in equality we have

$$\left|\frac{(-2)^n}{n^2} - L\right| \geq \frac{2^n}{n^2} - L$$

   Now since 2^n grows much faster than n^2, given any $N \in \mathbb{N}$, we can choose an even number $n > N$ large so that

$$\frac{2^n}{n^2} > L + 2$$

   and it follows

$$\left|\frac{(-2)^n}{n^2} - L\right| > L + 2 - L > 2 > 1$$

   a contradiction, so the limit must not be equal to L for any $L \in \mathbb{R}$. Hence the limit must not exist.

5. Without loss of generality we may suppose $\{x_n\}$ is nondecreasing (otherwise consider $\{-x_n\}$). Suppose for a contradiction that there is some $N > k$ such that $x_N > x_k$ (since the sequence is nondecreasing $x_N \geq x_k$). Let $\varepsilon = x_N - x_k$, it follows from monotonicity that

$$|x_n - x_k| = x_n - x_k \geq x_N - x_k > \varepsilon$$

   for every $n \geq N$, so x_k clearly cannot be the limit of the sequence, a contradiction.

6. (a) Let $I_n = [a_n, b_n]$. The nested condition ensures that $\{a_n\}$ is a monotone nondecreasing sequence. Note also that $\{a_n\}$ is bounded above by b_1, so by the monotone convergence theorem we have $a_n \to a$ for some $a \in \mathbb{R}$. We claim that $a \in \bigcap_{n=1}^{\infty} I_n$. Note that it suffices to show that $a \leq b_n$ for every n.

Suppose for a contradiction that $a > b_{n_0}$ for some n_0. Let $\varepsilon = a - b_{n_0}$. Since $a_n \to a$, there is $N \in \mathbb{N}$ such that $n > N$ implies $a - a_n < \varepsilon = a - b_{n_0}$. On the other hand, since the intervals are nested we have $a - b_{n_0} < a - b_n$ for every $n > n_0$. Thus choosing $n > \max\{N, n_0\}$ we have

$$a - a_n < \varepsilon = a - b_{n_0} < a - b_n \implies b_n < a_n$$

a contradiction, since the intervals are assumed to be nonempty. This proves $a \leq b_n$ for every n and so $a \in I_n$ for every n, which in turn implies $a \in \bigcap_{n=1}^{\infty} I_n$.

(b) Consider $J_n = (0, 1/n)$, then $\bigcap_{n=1}^{\infty} J_n = \emptyset$.