1. (a) Let $I = [a, b]$. Suppose for a contradiction that $x = \lim_{n\to\infty} x_n \notin I$. Without loss of generality we may assume $x > b$. Let $\varepsilon = x - b$. By definition of the limit there is $N \in \mathbb{N}$ such that $n > N$ implies $|x_n - x| < \varepsilon$, but this implies

$$x_n \in (x - \varepsilon, x + \varepsilon) = (b, x + \varepsilon)$$

which is disjoint from I, a contradiction.

(b) Consider $\left\{\frac{1}{n+1}\right\}_{n=1}^{\infty} \subset (0, 1)$, the limit is 0 as $n \to \infty$ which is not in $(0, 1)$.

2. Let $x = \lim_{n\to\infty} x_n$, we must show that $\lim_{n\to\infty} x_n^k = x^k$. Given $\varepsilon > 0$, choose $N_1 \in \mathbb{N}$ such that $n > N_1$ implies

$$|x_n - x| < \frac{\varepsilon}{k(|x| + 1)^{k-1}}$$

Since convergent sequences are bounded, we may choose $N_2 \in \mathbb{N}$ such that $n > N_2$ implies $|x_n| \leq |x|+1$. Let $N = \max\{N_1, N_2\}$. For $n > N$ we have

$$\begin{aligned}
\left|x_n^k - x^k\right| &= |x_n - x| \left|x_n^{k-1} + x x_n^{k-2} + x^2 x_n^{k-3} + \cdots + x^{k-1}\right| \\
&\leq |x_n - x| \left(|x_n|^{k-1} + |x| \, |x_n|^{k-2} + \cdots + |x|^{k-1}\right) \\
&\leq \frac{\varepsilon}{k(|x| + 1)^k} \left((|x| + 1)^{k-1} + |x| \, (|x| + 1)^{k-2} + \cdots + |x|^{k-1})\right) \\
&< \frac{\varepsilon}{k(|x| + 1)^k} (k(|x| + 1)^k) < \varepsilon
\end{aligned}$$

which gives the required result by definition.

3. (a) Let $M \in \mathbb{R}$ be such that $|a_n| \leq M$. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|b_n| < \frac{\varepsilon}{M}$$

Then for $n \geq N$ we have

$$|a_n b_n| \leq |a_n| \, |b_n| < M \frac{\varepsilon}{M} < \varepsilon$$

so $\lim_{n\to\infty} a_n b_n = 0$.

(b) Consider $a_n = n^2$ and $b_n = \frac{1}{n}$, then $a_n b_n = n \to \infty$ as $n \to \infty$.

(c) Let $a_n = 1$ if n is even and -1 if n is odd. Let $b_n = 1$. Then $a_n b_n = a_n$ which is divergent.

4. Let $\varepsilon > 0$. By definition, there is $N_1 \in \mathbb{N}$ such that $n > N_1$ implies $|x_{2n} - x| < \varepsilon$. Similarly there is N_2 such that $n > N_2$ implies $|x_{2n-1} - x| < \varepsilon$. Let $N = \max\{2N_1, 2N_2\}$ and clearly $n > N$ implies $|x_n - n| < \varepsilon$ (since n is either even or odd). So $\lim_{n\to\infty} x_n = x$ as well.

5. (a) $x_1 = 1$ is obvious. By recursive definition we have

$$x_{n+1} = \frac{f_{n+2}}{f_{n+1}} = \frac{f_{n+1} + f_n}{f_{n+1}} = 1 + \left(\frac{f_{n+1}}{f_n}\right)^{-1} = 1 + x_n^{-1}$$

(b) There are 5 things to show: $x_1 \geq 1$, $x_2 \leq 2$, $x_{2i-1} \leq x_{2i+1}$ for all $i \in \mathbb{N}$, $x_{2i} \leq x_{2i-2}$ for all $i \in \mathbb{N}$ and $x_{2i+1} \leq x_{2i}$ for all $i \in \mathbb{N}$. The first two are obvious. To show the third item we have by the recursive relation

$$x_{2i+1} = 1 + x_{2i}^{-1} = 1 + (1 + x_{2i-1}^{-1})^{-1}$$

so that

$$x_{2i-1} \leq x_{2i+1} \iff x_{2i-1}(x_{2i-1} + 1) \leq x_{2i-1} + 1 + x_{2i-1} \iff x_{2i-1}^2 \leq 1 + x_{2i-1}$$

which is true for $\frac{1-\sqrt{5}}{2} \leq x_{2i-1} \leq \frac{1+\sqrt{5}}{2}$.

Note that $x_1 = 1$ is between 0 and $\frac{1+\sqrt{5}}{2}$, so $x_1 \leq x_3$. More generally one can show that this relation is preserved under the recursive relation. Indeed

$$0 \leq x_{2i-1} \leq \frac{1+\sqrt{5}}{2} \implies x_{2i} = 1 + x_{2i-1}^{-1} \geq 1 + \frac{2}{1+\sqrt{5}} = \frac{1+\sqrt{5}}{2}$$
$$\implies x_{2i+1} = 1 + (1 + x_{2i-1}^{-1})^{-1} \leq 1 + \frac{2}{1+\sqrt{5}} = \frac{1+\sqrt{5}}{2}$$

This means $0 \leq x_{2i-1} \leq \frac{1+\sqrt{5}}{2}$ for all i and it follows from the previous paragraph that $x_{2i-1} \leq x_{2i+1}$.

The fourth item follows in a similar fashion: one shows that $x_{2i} \geq \frac{1+\sqrt{5}}{2}$ is preserved under the recursive relation and that

$$x_{2i-2} \geq x_{2i} \iff x_{2i-2} \geq 1 + x_{2i-2}^{-1} \iff x \geq \frac{1+\sqrt{5}}{2} \text{ or } x \leq \frac{1-\sqrt{5}}{2}$$

Finally the fifth item follows from the above since

$$x_{2i-1} \leq \frac{1+\sqrt{5}}{2} \leq x_{2i}$$

Combining all these gives the desired inequalities.

(c) We use Problem 4. By the recurrence relation we have that the odd-numbered terms satisfy

$$x_{2i+1} = 1 + x_{2i}^{-1} = 1 + (1 + x_{2i-1}^{-1})^{-1}$$

Now by part b) the odd-numbered term is a monotonically increasing sequence bounded above by 2. So by the monotone convergence theorem the limit exists, and one can directly solve the above equation to get

$$\lim_{n\to\infty} x_{2n-1} = \frac{1+\sqrt{5}}{2}$$

since by the previous part $x_{2n+1} \geq 1$ for all n. Similarly the even-numbered terms also have a limit, and a similar computation shows that this limit is also $\frac{1+\sqrt{5}}{2}$, so Problem 4 implies that

$$\lim_{n\to\infty} x_n = \frac{1+\sqrt{5}}{2}$$

6. Recall that

$$\limsup_{n\to\infty} = \lim_{n\to\infty} \sup_{k>n} x_k$$

Moreover we have

$$\sup_{k>n} x_k = -\inf_{k>n} (-x_k)$$

Combining these and noticing that the negative sign can be pulled out of the limit since the limit exists gives the result.

7. (a) This is an easy consequence of the triangle inequality.

(b) Recall that we have proved that (for example, Problem 8 in PS2, with the domain of f being $\mathbb{N} \setminus \mathbb{N}_n$)

$$\sup_{k>n}(x_k + y_k) \leq \sup_{k>n} x_k + \sup_{k>n} y_k$$

for every $n \in \mathbb{N}$. Taking limit as $n \to \infty$ both sides shows the desired result.

(c) Let x_n be 1 when n is even and -1 when n is odd, and y_n be 1 when n is odd and -1 when n is even. Clearly $x_n + y_n$ is identically 0 so $\limsup(x_n + y_n) = 0$. On the other hand it is clear that $\limsup x_n = \limsup y_n = 1$, so in this case

$$\limsup_{n\to\infty}(x_n + y_n) < \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$