1. Since the series is absolutely convergent, the right hand side is finite. Moreover, by the triangle inequality, the partial sums satisfy

$$\left|\sum_{n=1}^{N} x_n\right| \le \sum_{n=1}^{N} |x_n|$$

Taking $N \to \infty$ and noticing that

$$\lim_{N\to\infty}\left|\sum_{n=1}^{N} x_n\right| = \left|\lim_{N\to\infty}\sum_{n=1}^{N} x_n\right|$$

(since the limit of the right hand side exists) gives the inequality.

2. (a) Let $\varepsilon > 0$. Since $a_n \to A$ there is $N \in \mathbb{N}$ such that $n > N$ implies

$$|a_n - A| < \varepsilon/2$$

Since a_n is convergent there is $M > 0$ such that $|a_n| \le M$ for all n. Now given $n > N$ we have by triangle inequality

$$|b_n - A| \le \sum_{k=1}^{n} \frac{1}{n}|a_n - A| \le \sum_{k=1}^{N} \frac{1}{n}|a_n - A| + \sum_{k=N}^{n} \frac{1}{n}|a_n - A| \le \sum_{k=1}^{N} \frac{2M}{n} + \sum_{k=N}^{n} \frac{\varepsilon}{2n} < \frac{2MN}{n} + \frac{\varepsilon}{2}$$

Now let N_1 be such that

$$\frac{2MN}{N_1} < \varepsilon/2$$

It follows that for $n > \max\{N, N_1\}$ we have

$$|b_n - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which means $b_n \to A$ as well.

(b) This follows immediately by taking $a_n = s_n$ in part (a).

(c) The partial sums are $s_k = 0$ if k is even and -1 if k is odd. So we have

$$\sum_{k=1}^{n} s_k = \begin{cases} -\frac{n}{2} & n \text{ even} \\ -\frac{n+1}{2} & n \text{ odd} \end{cases}$$

and it follows that

$$\frac{1}{n}\sum_{k=1}^{n} s_k = \begin{cases} -\frac{1}{2} & n \text{ even} \\ -\frac{1}{2} - \frac{1}{2n} & n \text{ odd} \end{cases}$$

In either case the limit is $-\frac{1}{2}$ as $n \to \infty$, so the series is Cesàro summable with $S = -\frac{1}{2}$

3. By definition $(c-\varepsilon, c+\varepsilon) \setminus \{c\} \cap A$ is not empty for every $\varepsilon > 0$, but clearly $(c-\varepsilon, c+\varepsilon) \setminus \{c\} \cap A \subset (c-\varepsilon, c+\varepsilon) \setminus \{c\} \cap S$, so the latter set is also not empty for any $\varepsilon > 0$.

4. Let $\varepsilon > 0$ by definition there is $\delta_1 > 0$ such that $|c - c_2| < \delta_1 \implies |g(c) - g(c_2)| < \varepsilon$. By definition again there is $\delta_2 > 0$ such that $|c - c_1| < \delta_2 \implies |f(c) - c_2| < \delta_1$. Let $\delta = \delta_2$ and for $|c - c_1| < \delta$ we have

$$|g(f(c)) - L| = |g(f(c)) - g(c_2)| < \varepsilon$$

since $|f(c) - c_2| < \delta_1$.

5. Since c is a cluster point of S, there is $x_n \in (c - 1/n, c + 1/n) \setminus \{c\}$ for every $n \in \mathbb{N}$. Clearly $x_n \to c$. Since f is bounded, by Bolzano-Weierstrass we may extract a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{f(x_{n_i})\}$ converges. Since $x_n \to c$ it follows $x_{n_i} \to c$ as well, so the sequnce $\{x_{n_i}\}$ is what we wanted.

6. Let $x_0 \in (0, \infty)$ and $\epsilon > 0$. We have

$$\left| \frac{1}{x} - \frac{1}{x_0} \right| = \frac{|x - x_0|}{x x_0}$$

Let us first agree that $|x - x_0| < \frac{1}{2} |x_0|$, so that $x > \frac{x_0}{2}$ and

$$\frac{|x - x_0|}{x x_0} < \frac{2 |x - x_0|}{x_0^2}$$

If further $|x - x_0| < \frac{\varepsilon x_0^2}{2}$ we will have

$$\frac{2 |x - x_0|}{x_0^2} < \varepsilon$$

Thus it suffices to pick $\delta = \min\{\frac{1}{2} |x_0|, \frac{\varepsilon |x_0|^2}{2}\}$ (so that both of the above inequalities hold).

7. By definition there is $\delta > 0$ such that $|x - c| < \delta$ implies

$$|f(x) - f(c)| < \frac{|f(c)|}{2}$$

which in turn (by triangle inequality) implies $f(x) > f(c) - \frac{f(c)}{2} = \frac{f(c)}{2} > 0$ for $|x - c| < \delta$.

8. Let $\varepsilon > 0$ and fix $x \in \mathbb{R}$. Let $\delta = \frac{\varepsilon}{L}$, then we have $|x - y| < \delta$ implies

$$|f(x) - f(y)| \leq L |x - y| < L \frac{\varepsilon}{L} = \varepsilon$$

which shows the continuity at x. Since x is arbitrary, f is continuous.