1. Let $\varepsilon > 0$. Since $f$ is Riemann integrable, there is a partition $P_1 = \{x'_0, \dots, x'_{n_1}\}$ such that

$$U(P_1, f) - L(P_1, f) < \frac{\varepsilon}{2}$$

(This follows by choosing a partition $Q_1$ such that $U(Q_1, f) - \inf_P U(P, f) < \varepsilon/2$ and a partition $Q_2$ such that $\sup_P L(P, f) - L(Q_2, f) < \varepsilon/2$, then take a common refinement of $Q_1$ and $Q_2$). This means for any $x_k^{'*} \in [x'_{k-1}, x'_k]$ we have

$$U(P_1, f) - \sum_{k=1}^{n_1} f(x_k^{'*})\Delta x'_k < U(P, f) - L(P_1, f) < \frac{\varepsilon}{2}$$

Since $f$ is Riemann integrable there is another partition $P_2$ such that

$$\left| U(P_2, f) - \int_a^b f(x)dx \right| < \frac{\varepsilon}{2}$$

Let $P = \{x_0, \dots, x_n\}$ be a common refinement of $P_1$ and $P_2$, then we have by triangle inequality

$$\left| \int_a^b f(x)dx - \sum_{k=1}^{n} f(x_k^*)\Delta x_k \right| < \left| \int_a^b f(x)dx - U(P, f) \right| + \left| U(P, f) - \sum_{k=1}^{n} f(x_k^*)\Delta x_k \right| < \varepsilon$$

2. Since $f$ is continuous on a bounded interval, we can bound $f$ by $f(x_1) \leq f(x) \leq f(x_2)$ for some $x_1, x_2 \in [a, b]$. Therefore appealing to Riemann sum we have

$$f(x_1)(b - a) \leq \int_a^b f(x)dx \leq f(x_2)(b - a)$$

Intermediate value theorem applied to the function $g(x) = f(x)(b-a)$ says that there is $c \in [x_1, x_2]$ (or $[x_2, x_1]$, depending on the order) such that

$$f(c)(b - a) = \int_a^b f(x)dx$$

3. Split the interval as $[a, c]$ and $[c, b]$. Consider $c_n = c - \frac{1}{n}$ (for $n$ sufficiently large). Then since $f = g$ on $[a, c_n]$ we have that $g$ is integrable on $[a, c_n]$ with

$$\int_a^{c_n} f(x)dx = \int_a^{c_n} g(x)dx$$

Now by Lemma 5.2.8 we may pass to the limit to conclude that $g$ is integrable on $[a, c]$ and that

$$\int_a^c g(x)dx = \lim_{n\to\infty} \int_a^{c_n} g(x)dx = \lim_{n\to\infty} \int_a^{c_n} f(x)dx = \int_a^c f(x)dx$$

Doing the same thing on $[c, b]$ and adding up the integral gives the required result.

4. Since $f$ is continuous, so is $|f|$, so we have that $|f|$ is Riemann integrable and that (since $f \leq |f|$)

$$\left| \int_a^b f(x)dx \right| \leq \left| \int_a^b |f(x)|\, dx \right| = \int_a^b |f(x)|\, dx$$

When $f$ is only Riemann integrable, the problem becomes much harder since we do not know if $|f|$ is Riemann integrable (so one has to work directly with Riemann sums to prove that $|f|$ is Riemann integrable).

5. Since $f$ is monotonically increasing on $[a, b]$ we have that $f(a) \leq f(x) \leq f(b)$ for $x \in [a, b]$. Now let $\varepsilon > 0$, and consider a uniform partition $P_n$ of length $\frac{b-a}{n}$ of $[a, b]$ (so that $x_0 = a$, $x_1 = a + \frac{b-a}{n}$, etc.). Then we have by monotonicity

$$U(P_n, f) - L(P_n, f) = \sum_{k=1}^{n} (f(x_k) - f(x_{k-1}))\frac{b-a}{n} \leq \frac{b-a}{n} \sum_{k=1}^{n} f(x_k) - f(x_{k-1}) = \frac{b-a}{n}(f(b) - f(a))$$

since the sum telescopes. It follows that if we choose $n$ sufficiently large we have $U(P_n, f) - L(P_n, f) < \varepsilon$, so $f$ is Riemann integrable.

6. (a) First we show that $f(x) = 0$ on $(a, b)$. Suppose for a contradiction that there is some $x_0 \in (a, b)$ such that $f(x_0) \neq 0$. Without loss of generality we may assume $f(x_0) > 0$. Since $f$ is continuous, there is $\delta > 0$ such that $|x - x_0| < \delta \implies f(x) > \frac{f(x_0)}{2}$. This means that

$$\int_a^b f(x)^2 dx \geq \int_{x_0-\delta}^{x_0+\delta} f(x)^2 dx > 2\delta \frac{f(x_0)^2}{4} > 0$$

a contradiction.

Now suppose again for a contradiction that $f(a) > 0$. The same idea works except one can only use a one-sided interval, i.e. there is $\delta > 0$ such that $x - a < \delta \implies f(x) > \frac{f(a)}{2}$. The rest is very similar.

(b) We first show that $f(x) = 0$ on $(a, b)$. Suppose for a contradiction (as above) that there is $x_0 \in (a, b)$ such that $f(x_0) > 0$. Again there is $\delta$ such that $|x - x_0| < \delta \implies f(x) > \frac{f(x_0)}{2}$. Let $\phi$ be a function that agrees with $f$ on $(x_0 - \delta/2, x_0 + \delta/2)$, linear on $(x_0 - \delta, x_0 - \delta/2)$ and $(x_0 + \delta/2, x_0 + \delta)$ with $\phi(x_0 - \delta) = \phi(x_0 + \delta) = 0$, and 0 elsewhere. It is easy to see that $\phi$ is continuous, and that

$$\int_a^b f(x)\phi(x) dx \geq \int_{x_0-\delta/2}^{x_0+\delta/2} f(x)^2 dx > \delta \frac{f(x_0)^2}{4} > 0$$

a contradiction.

Now suppose for a contradiction that $f(a) > 0$. Again we find $\delta$ such that $x - a < \delta \implies f(x) > \frac{f(a)}{2}$. Let $\phi$ be a function that agrees with $f$ on $[a + \delta/4, a + \delta/2]$, linear on $[a, a + \delta/4)$ and $(a + \delta/2, a + \delta]$ with $\phi(a) = \phi(a + \delta) = 0$, and 0 elsewhere. Again $\phi$ is continuous and

$$\int_a^b f(x)\phi(x) dx \geq \int_{a+\delta/4}^{a+\delta/2} f(x)^2 dx > \frac{\delta}{4} \frac{f(a)^2}{4} > 0$$

a contradiction. So $f$ must be identically 0.

7. This follows from the fundamental theorem of calculus and the product rule, i.e.

$$\int_a^b F(x)G'(x) + F'(x)G(x) dx = \int_a^b (F(x)G(x))' dx = F(b)G(b) - F(a)G(a)$$

8. By the fundamental theorem of calculus we have for $x \in (0, 2)$,

$$f(x) - f(0) = \int_0^x f'(y) dy \geq \int_0^x y dy = \frac{1}{2}x^2$$

which implies $f(x) \geq \frac{1}{2}x^2$ since $f(0) = 0$.

On the other hand for $x \in (-2, 0]$ we have

$$f(0) - f(x) = \int_x^0 f'(y) dy \geq \int_x^0 y dy = -\frac{1}{2}x^2$$

So we get $f(x) \leq \frac{1}{2}x^2$ instead.