1. Recall that

$$\|f\|_u = \sup\{|f(x)| \mid x \in S\}$$

   (a) By triangle inequality we have

   $$|f(x) + g(x)| \leq |f(x)| + |g(x)|$$

   Taking supremum both sides and using

   $$\sup_{x\in S}(|f(x)| + |g(x)|) \leq \sup_{x\in S}|f(x)| + \sup_{x\in S}|g(x)|$$

   gives the inequality.

   (b) Clearly

   $$|f(x)g(x)| = |f(x)|\,|g(x)|$$

   Taking supremum both sides and using

   $$\sup_{x\in S}|f(x)|\,|g(x)| \leq \left(\sup_{x\in S}|f(x)|\right)\left(\sup_{x\in S}|g(x)|\right)$$

   (since $|f|$ and $|g|$ are nonnegative) gives the inequality.

2. (a) Let $\varepsilon > 0$ and $x \in S$. Since $f_n \to f$ pointwise there is $N_1 \in \mathbb{N}$ such that $n > N_1$ implies $|f_n(x) - f(x)| < \varepsilon/2$. Similarly there is $N_2 \in \mathbb{N}$ such that $n > N_2$ implies $|g_n(x) - g(x)| < \varepsilon/2$. Let $N = \max\{N_1, N_2\}$ and $n > N$ implies

   $$|f_n(x) + g_n(x) - f(x) - g(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \varepsilon$$

   so $f_n + g_n = h_n \to f + g$ pointwise.

   (b) The exact same argument as above works. Instead one does not choose a specific $x \in S$ at the first place to account for uniform convergence.

3. Suppose for a contradiction that there is $x, y \in (a, b)$ with $x < y$ such that $f(y) < f(x)$. Let $\varepsilon = f(x) - f(y) > 0$. Since $f_n \to f$ pointwise, there is $N_1$ such that $n > N_1$ implies

$$|f_n(x) - f(x)| < \varepsilon/3 \implies f_n(x) > f(x) - \varepsilon/3$$

Similarly there is $N_2$ such that $n > N_2$ implies

$$|f_n(y) - f(y)| < \varepsilon/3 \implies f_n(y) < f(y) + \varepsilon/3$$

Let $N = \max\{N_1, N_2\}$ and $n > N$ implies that

$$f_n(y) - f_n(x) < f(y) + \varepsilon/3 - f(x) + \varepsilon/3 = -\varepsilon + 2\varepsilon/3 = -\varepsilon/3 < 0$$

a contradiction since $f_n$ is non-decreasing.

4. Let $\varepsilon > 0$. Since $f$ is uniformly continuous, there is $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Choose $N$ such that $\frac{1}{N} < \delta$, then $n > N$ implies

$$|f_n(x) - f(x)| = |f(x + 1/n) - f(x)| < \varepsilon$$

so $f_n \to f$ uniformly.

5. (a) Since $h$ is continuous on $[a, b]$, $h$ is uniformly continuous on $[a, b]$ (bounded interval implies uniform continuity), therefore $f$ is also uniformly continuous (since it is constant outside of $[a, b]$). $f$ is clearly bounded.

(b) By the fundamental theorem of calculus we have

$$f_n'(x) = \frac{n}{2}(f(x+1/n) - f(x-1/n))$$

which is uniformly continuous (since composition of uniformly continuous functions is still uniformly continuous). Moreover,

$$|f_n(x)| \leq \frac{n}{2}(x+1/n-(x-1/n)) \sup_{x\in(x-1/n,x+1/n)} |f(x)| \leq \|f\|_u$$

So $\|f_n\|_u \leq \|f\|_u$.

(c) Let $\varepsilon > 0$. Since $f$ is uniformly continuous, there is $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Choose $N$ such that $1/N < \delta$. For $n > N$ we have

$$\begin{aligned}|f_n(x) - f(x)| = \frac{n}{2}\left|\int_{x-1/n}^{x+1/n} f(t)dt - \int_{x-1/n}^{x+1/n} f(x)dt\right| \leq \frac{n}{2}\int_{x-1/n}^{x+1/n} |f(x) - f(t)|\, dt \\ \leq \frac{n}{2}\int_{x-1/n}^{x+1/n} \varepsilon dt = \frac{n}{2}\cdot\frac{2}{n}\varepsilon = \varepsilon\end{aligned}$$

(since $|t-x| < 1/n < \delta$ for $t \in (x-1/n, x+1/n)$) so $f_n \to f$ uniformly on $\mathbb{R}$. In particular, $f_n \to h$ uniformly on $[a,b]$.

6. (a) The function clearly has no issues for $x \neq 0$. We need to check that $\lim_{x\to 0} P(1/x)\phi(x) = 0$ for any polynomial $P$. By linearity of the limit it suffices to show for $P(x) = x^n$ for $n \in \mathbb{N}\cup\{0\}$.

We use induction. This is clearly true for $n = 0$. Suppose the claim is true for all $n = k$, then for $n = k+1$ by L'Hopital's rule we have

$$\lim_{x\to 0} x^{-(k+1)}e^{-1/x} = \lim_{x\to 0}\frac{x^{-(k+1)}}{e^{1/x}} = \lim_{x\to 0}\frac{-(k+1)x^{-(k+2)}}{-x^{-2}e^{1/x}} = \lim_{x\to 0}\frac{(k+1)x^{-k}}{e^{1/x}} = 0$$

Hence by induction we have $\lim_{x\to 0} x^{-n}e^{-1/x} = 0$ for all $n \in \mathbb{N}$. Thus $f(x)$ is continuous.

(b) We use induction again. For $k = 0$ the statement is clearly true. Suppose $\phi^{(k)}(x) = P_k(1/x)\phi(x)$ for some polynomial $P_k$. For $x > 0$ using the chain rule we have

$$\phi^{(k+1)}(x) = P_k'(1/x)\cdot(-\frac{1}{x^2})\phi(x) + (\frac{1}{x^2}P_k(1/x))\phi(x) = ((-x^2P_k')(1/x) + (x^2P_k)(1/x))\phi(x)$$

which is again of the form $P_{k+1}(1/x)\phi(x)$, where

$$P_{k+1}(x) = -x^2P_k' + x^2P_k$$

is again a polynomial. For $x < 0$ clearly $\phi^{(k+1)}(x) = P_{k+1}(1/x)\phi(x)$ still holds since $\phi(x)$ is identically 0. Finally we have

$$\lim_{x\to 0}\phi^{(k+1)}(x) = 0$$

by part (a). So $\phi^{(k+1)}(x)$ is a continuous function of the form $\phi^{(k+1)}(x) = P_{k+1}(x)\phi(x)$ for all $x \in \mathbb{R}$. Consequently it is a smooth function on $\mathbb{R}$ by part (a).

7. (a) Consider $\psi(x) = \phi((x-a)(b-x))$ for $x \in (a,b)$. Clearly when $x \leq a$ or $x \geq b$ we have $\psi(x) = 0$, and when $x \in (a,b)$ we have $\psi(x) > 0$ sicne $(x-a)(b-x) > 0$. Moreover $\psi$ is smooth since $\phi$ is.

(b) Let

$$C = \int_a^b \psi(x)dx$$

(which can be computed explicitly). Now define

$$\eta(x) = \frac{1}{C}\int_{-\infty}^x \psi(t)dt$$

Clearly $\eta$ is again a smooth function. Moreover when $x < a$, $\eta(x) = 0$ since $\psi(x) = 0$, and when $x > b$ we have

$$\eta(x) = \frac{1}{C} \int_{-\infty}^{b} \psi(t)dt = \frac{1}{C} \int_{a}^{b} \psi(t)dt = 1$$

Finally when $x \in (a, b)$ we must have $0 \leq \eta(x) \leq 1$ since $\eta(x)$ is clearly increasing.

(c) Let $\psi_1(x)$ be the function as constructed in part (b) such that $\psi_1(x) = 0$ for $x \leq a$ and $x = 1$ for $x \geq c$. Let $\psi_2(x)$ be the function as constructed in part (b) such that $\psi_2(x) = 0$ for $x \leq d$ and $\psi_2(x) = 1$ for $x \geq b$. Consider $\zeta(x) = \psi_1(x) - \psi_2(x)$ which is clearly smooth. For $x \leq a$ we have $\psi_1(x) = \psi_2(x) = 0$. For $a \leq x \leq c$ we have $\zeta(x) = \psi_1(x)$ which is between 0 and 1. For $c \leq x \leq d$ we again have $\zeta(x) = \psi_1(x)$ which is identically 1. For $d \leq x \leq b$ we have $\zeta(x) = 1 - \psi_2(x)$ which is again between 0 and 1. Finally for $x \geq b$ we have $\zeta(x) = 1 - 1 = 0$ again. So this is precisely what we want.