Final Exam Solutions

1. Let $f(x)$ be a continuous function on [a,b] with $f(a) < 0 < f(b)$.
a. (10 pts) Let x_0 be such that $f(x_0) > 0$. Show that there is a interval of the form $I = (x_0 - \delta, x_0 + \delta)$ such that $f(x) \geq \frac{f(x_0)}{2}$ on $I \cap (a, b)$.

By the uniform continuity of f with $\varepsilon = \frac{f(x_0)}{2}$, there exists $\delta = \delta(\varepsilon)$ such that $|f(x) - f(x_0)| < \varepsilon$ if $x \in I \cap (a, b)$. Hence

$$f(x) \geq f(x_0) - \varepsilon = \frac{f(x_0)}{2} \quad \text{if } x \in I \cap (a, b) \; .$$

b. (10pts) Let $S = \{x \in [a, b] : f(x) > 0\}$ and define $c = \inf S$. Show that $f(c) = 0$.

S is bounded below so c exists. (Note that S need not be connected but this does not matter.) By the continuity of f, c is a point of $\overline{S}$ so we have $f(c) \geq 0$. If $f(c) > 0$ then by the Intermediate value theorem, there is a point $x_0 \in (a, c)$ where $0 < f(x_0) < f(c)$ contradicting the definition of c. Thus $f(c) = 0$.

2. (20pts) Let $f(x)$ be a function which is differentiable on (-1,1) and continuous on [-1,1]. Suppose also that $f'(x) \geq 0$ for $x \in (-1, 0]$ and $f'(x) \leq 0$ for $x \in [0, 1)$. Show that $f(x)$ has its global maximum at $x = 0$. Justify.

Clearly $f'(0) = 0$. By the mean value theorem if $x \in [-1, 0]$, then $f(x) - f(0) = xf'(c) \leq 0$ since $c \in (x, 0)$ so $f'(c) \geq 0$. Similarly if $x \in (0, 1]$, $f(x) - f(0) = xf'(c) \leq 0$ since $f'(c) \leq 0$. Hence $f(x)$ has its global max at $x = 0$.

3. Let $f(x)$ be defined for $x > 0$ by

$$f(x) = \int_1^x \frac{dt}{t} \; .$$

a. (10pts) Compute the formal Taylor series of $f(x)$ about $x = 1$.

By the fundamental theorem I (since the integrand $\frac{1}{t}$ is continuous for $t > 0$), $f(x)$ is C^1 and $f'(x) = \frac{1}{x}$. Inductively, $f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}$ and $f(1) = 0$, $f^{(n)}(1) = (-1)^{n-1}(n-1)!$, $n \geq 1$. So formally

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1}(n-1)! \frac{(x-1)^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \; .$$

b. (10pts) Find the radius of convergence of this series and justify.

We have $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \limsup_{n\to\infty} (\frac{1}{n})^{\frac{1}{n}} = 1$ since $\log \frac{1}{R} = \lim_{t\to 0} t \log t = 0$.

4. (20pts) State the Cauchy criterion for Riemann integrability and use it to show that any continuous function f on [0,1] is Riemann integrable. You may state and use any of the basic properties of continuous functions on compact intervals.

The Cauchy criterion states that a function f on [a,b] is Riemann integrable if given any $\varepsilon > 0$, there is a partition P of [a,b] such that $S^+(f,P) - S^-(f,P) \le \varepsilon$ where

$$S^+(f,P) = \sum M_i(x_i - x_{i-1}),\ S^-(f,P) = \sum m_i(x_i - x_{i-1}),\ M_i = \sup_{[x_{i-1},x_i]} f,\ m_i = \inf_{[x_{i-1},x_i]} f\ .$$

Let $\varepsilon > 0$ be given. Since $f(x)$ is uniformly continuous (a continuous function on a compact set is uniformly continuous), there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta,\ x, y \in [0,1]$. Choose a partition $x_i = \frac{i}{n}$, $i = 0, 1, \ldots, n$ with $\frac{1}{n} < \delta$. Then $M_i - m_i < \varepsilon$ since $x_i - x_{i-1} = \frac{1}{n} < \delta$. Hence

$$S^+(f,P) - S^-(f,P) \le \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < \varepsilon \sum_{i=1}^{n} (x_i - x_{i-1}) = \varepsilon\ .$$

Hence f is Riemann integrable on [a,b].

5a. (10pts) Define what it means for a family of functions $\mathcal{F}$ defined on $[a,b]$ to be equicontinuous.

A family $\mathcal{F}$ of functions defined on a common domain D (usually an interval) is said to be equicontinuous if given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that $|f(x) - f(y)| < \varepsilon$ if $|x - y| < \delta$ for all $f \in \mathcal{F}$.

5b. (10pts) Let N be a fixed positive integer and define $\mathcal{F}$ to be the family of all polynomials $p(x) = \sum_{j=0}^{N} c_j x^j$ where $|c_j| \le 1$. Show that $\mathcal{F}$ is equicontinuous on any [a,b]. Hint: What can you say about $|p'(x)|$?

Observe that for any $p(x) \in \mathcal{F}$, $|p'(x)| \le \sum_{j=1}^{N} jM^{j-1} \le C$ where C depends only on N, a, b. In particular by the mean value theorem $|p(x) - p(y)| \le C|x - y| \quad \forall x, y \in [a,b]$. This implies equicontinuity with $\delta = \frac{\varepsilon}{C}$.

6a. (10pts) Define what it means for the series $\sum_{n=1}^{\infty} a_n$ to converge.

The series $\sum_{n=1}^{\infty} a_n$ converges if the sequence of partial sums $s_N = \sum_{n=1}^{N}$ converges.

6b. (10pts) Show that if $a_n \geq 0$ $\forall n$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^p$ converges for all $p > 1$.

If $\sum_{n=1}^{\infty} a_n$ converges, then necessarily $a_n \to 0$ so we may choose N such that $0 \leq a_n \leq \frac{1}{2}$ if $n \geq N$. In particular $0 \leq a_n^p < a_n$ for $n \geq N$. Therefore the series $\sum_{n=1}^{\infty} a_n^p$ converges since its partial sums (starting with $n = N$) are increasing and bounded.