Solutions Midterm Exam 2 — Apr. 9, 2014

1. Determine whether the following statements are true or false. Justify your answer (i.e., prove the claim, derive a contradiction or give a counter-example).

   (a) (10 points) If $f : D \to \mathbb{R}$ is continuous and D is closed, then for all $C \subset \mathbb{R}$ closed, $f^{-1}(C)$ is closed.

   True. Let $x_n \in f^{-1}(C)$ satisfy $x_n \to x \in \mathbb{R}$ as $n \to \infty$. As D is closed, $x \in D$. Hence, as f is continuous, $f(x_n) \to f(x)$ and since C is closed $f(x) \in C$. Hence, $x \in f^{-1}(C)$. As this is true of all convergent sequences, $f^{-1}(C)$ is closed.

   (b) (10 points) If $f : D \to \mathbb{R}$ is continuous, and $D \subset \mathbb{R}$ is closed, then $f(D)$ is closed.

   False. Let $D = \mathbb{R}$, which is closed and $f(x) = \frac{1}{1+x^2}$, which is continuous. Then $f(\mathbb{R}) = (0, 1]$ which is not closed.

(c) (10 points) If $f : (a, b) \to \mathbb{R}$ is C^1 and injective, then $f' \neq 0$.

False. Consider $f(x) = x^3$ on $(-1, 1)$.

(d) (10 points) There is no differentiable function $f : (-1, 1) \to \mathbb{R}$ with $f'(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$.

True. If there was such a differentiable f, then the fact that $f'(-1/2) = -1$ and $f'(1/2) = 1$ together imply the existence of a $z \in (-1/2, 1/2)$ so that $f'(z) = 0$.

(e) (10 points) Suppose $f : (-1, 1) \to \mathbb{R}$ is C^1 and $f(0) = 0$. If $f' = O(|x|), x \to 0$, then $f = O(|x|^2), x \to 0$.

True. For $x \neq 0$, the mean value theorem implies that $\frac{f(x)}{x} = \frac{f(x)-f(0)}{x-0} = f'(x_1)$ for some x_1 between 0 and x. That is, $|f(x)| = |f'(x_1)||x|$ for some x_1 with $0 < |x_1| < |x|$. As $f' = O(|x|), x \to 0$, there is a $\delta > 0$ and a $C > 0$ so if $|x| < \delta$, then $|f'(x)| \leq C|x|$. Hence, for $|x| < \delta$, $|f(x)| \leq C|x_1||x| \leq C|x|^2$. That is, $f = O(|x|^2), x \to 0$.

(f) (10 points) If $f : (-1, 1) \to \mathbb{R}$ is C^3 and has Taylor polynomial at $x_0 = 0$ given by $T_3(f, 0; x) = 3 + x^2 - 100x^3$, then f has a strict local minimum at $x_0 = 0$.

True. $f'(0) = 0$ and $f''(0) = 2 > 0$ and so $x_0 = 0$ is a strict local minimum.

2. (15 points) Let $f : (a, b) \to \mathbb{R}$ be uniformly continuous. Show that $\lim_{x \to b} f(x)$ exists.

Let $x_k \in (a, b)$ satisfy $x_k \to b$. By the uniform continuity, $y_k = f(x_k)$ is a Cauchy sequence. Indeed, for each $\epsilon > 0$, there is a $\delta > 0$ so that if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Hence, if m is choosen so that $m < n$ implies $|x_n - b| < \delta/2$, then if $m < n, k$, then $|x_n - x_k| < \delta$ and hence $|y_n - y_k| < \epsilon$. Let $L = \lim_{k\to\infty} y_k$. We show that $|\lim_{x\to b} f(x) = L$. Indeed, for each $\epsilon > 0$, choose $\delta > 0$ as before. Pick some $x_k \in (b - \delta, b)$ so that $|f(x_k) - L| < \epsilon$ (such x_k exists as $x_k \to b$ and $f(x_k) \to L$). For any $x \in (b - \delta, b)$, $|x - x_k| < \delta$ and hence $|f(x) - L| \leq 2\epsilon$ by the triangle inequality.

3. (a) (5 points) Show that for any pairs (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$, there is a unique affine function g with $g(x_i) = y_i$, $i = 1, 2$.

Set $g(x) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1$. This is a well-defined affine function with the desired properties. On the other, hand if $g(x) = mx + b$ is affine, then one sees that $g(x_i) = y_i$ only if $\frac{y_2 - y_1}{x_2 - x_1} = m$ and $y_1 = b$ which shows g is unique.

(b) (10 points) Let $f : (a, b) \to \mathbb{R}$ be C^2 and suppose that $f''(x) < 0$ for all $x \in (a, b)$. Show that if g is an affine function with $g(x_1) = f(x_1)$ and $g(x_2) = f(x_2)$, for $a < x_1 < x_2 < b$, then $g(x) < f(x)$ for all $x \in (x_1, x_2)$.

Set $h(x) = f(x) - g(x)$. So $h(x_1) = h(x_2) = 0$. If there was any point $z \in (x_1, x_2)$ so that $h(z) \leq 0$, then h would have a local minimum at some point $z' \in (x_1, x_2)$. At such a point $h''(z') \geq 0$. However, $h''(z') = f''(z') < 0$, so this is not possible. Hence, $h(x) > 0$ for all $x \in (x_1, x_2)$.

(c) (10 points) Let $f : (a, b) \to \mathbb{R}$ be C^2. Show that if, for all $a < y < z < b$, $f\left(\frac{y+z}{2}\right) \le \frac{f(y)+f(z)}{2}$, then $f''(x) \ge 0$ for all $x \in (a, b)$. (Hint: If g is affine, then $g\left(\frac{y+z}{2}\right) = \frac{g(y)+g(z)}{2}$).

Suppose $f''(c) < 0$ for some $c \in (a, b)$. By the continuity of f'', there is a neighborhood U of c so that $f'' < 0$. Pick $y, z \in U$ so $(y, z) \subset U$. If g is the affine function with $g(y) = f(y)$ and $g(z) = f(z)$, then part (b) implies that $f(x) > g(x)$ for all $x \in (y, z)$. Hence, $f\left(\frac{y+z}{2}\right) > g\left(\frac{y+z}{2}\right) = \frac{g(y)+g(z)}{2} = \frac{f(y)+f(z)}{2}$. This contradicts our assumption and proves the result.