Math 405 Final Exam — May. 14, 2014

Name: ______________________________________

- Complete the following problems. In order to receive full credit, please make sure to *justify your answers*. You are free to use results from class or the course textbook as long as you clearly state what you are citing.
- **You have 3 hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted (nor are they needed). If you finish early, you must hand your exam paper to a proctor.
- Please check that your copy of this exam contains 10 numbered pages and is correctly stapled.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

The following boxes are strictly for grading purposes. Please do not mark.

Question	Points	Score
1	80	
2	25	
3	25	
4	35	
5	35	
Total	200	

1. Determine whether the following statements are true or false. Justify your answer (i.e., prove the claim, derive a contradiction or give a counter-example).

   (a) (10 points) There exist open intervals I_n with $I_{n+1} \subset I_n$ so that $\cap_{n=1}^{\infty} I_n = \emptyset$.

   (b) (10 points) If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous and $\{x_n\}$ is Cauchy, then $\{f(x_n)\}$ is Cauchy.

(c) (10 points) If $f : (a, b) \to \mathbb{R}$ is C^1 and strictly increasing, then $f'(x) > 0$ for each $x \in (a, b)$.

(d) (10 points) If $f : (-1, 1) \to \mathbb{R}$ is C^2 with $f(0) = f'(0) = 0$ and $f''(0) = 2$, then there is an interval I containing 0 so that $f(x) \geq 0$ for $x \in I$.

(e) (10 points) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

(f) (10 points) There is a sequence of continuous functions $f_n : [-1,1] \to \mathbb{R}$ converging uniformly to the function $f : [-1,1] \to \mathbb{R}$ given by $f(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$.

(g) (10 points) If $\sum_{n=1}^{\infty} a_n$ is a convergent series, then for all bijections $m : \mathbb{N} \to \mathbb{N}$ the series $\sum_{n=1}^{\infty} a_{m(n)}$ is convergent and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{m(n)}$.

(h) (10 points) Let $f : [0,1] \to \mathbb{R}$ be Riemann integrable. If $f(q) = 0$ for all rational numbers $q \in [0,1]$, then $\int_0^1 f(x)dx = 0$.

2. (a) (10 points) State the intermediate value theorem.

   (b) (15 points) Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $I \subset \mathbb{R}$ is a compact interval, then $f(I)$ is compact interval.

3. (a) (10 points) State the mean value theorem.

(b) (15 points) Show that if $f : \mathbb{R} \to \mathbb{R}$ is *differentiable* and $f'(x) \geq x$, then $f(x) \leq f(0) + \frac{1}{2}x^2$ when $x \leq 0$.

4. (a) (10 points) State one of the (equivalent) definitions of a function $f : [a, b] \to \mathbb{R}$ being Riemann integrable.

   (b) (10 points) Give an example of a function $f : [0, 1] \to \mathbb{R}$ which is not Riemann integrable. You do not need to justify this.

(c) (15 points) Using the definition from a), show that if $f : [a, b] \to \mathbb{R}$ is continuous, then it is Riemann integrable.

5. Let $f : D \to \mathbb{R}$ be a function.

   (a) (10 points) State the definition of f being (real) analytic.

   (b) (10 points) Give an example of a function f that is infinitely differentiable (i.e. of class C^∞) but that is not real analytic. You do not need to justify your answer.

(c) (15 points) Show that if D is an interval, f is real analytic and $f(x) = 0$ for all $x \in I$ for $I \subset D$ an open interval, then $f(x) = 0$ for all $x \in D$.

Hint: Consider the maximum interval containing I on which f vanishes. Using the Taylor polynomials at the endpoints prove this interval is D.

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