

Math 405 Midterm Exam 1 — Mar. 5, 2014

Name: _____

- Complete the following problems. In order to receive full credit, please make sure to *justify your answers*. You are free to use results from class or the course textbook as long as you clearly state what you are citing.
- **You have 75 minutes.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted (nor are they needed). If you finish early, you must hand your exam paper to a proctor.
- Please check that your copy of this exam contains 5 numbered pages and is correctly stapled.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

The following boxes are strictly for grading purposes. Please do not mark.

Question	Points	Score
1	60	
2	20	
3	20	
Total	100	

1. Determine whether the following statements are true or false. Justify your answer (i.e., prove the claim, derive a contradiction or give a counter-example).

(a) (10 points) If $A \subset B$, and B is countable, then A is countable.

(b) (10 points) If \mathcal{B} is an open cover of $(0, 1]$, then \mathcal{B} has a finite subcover.

(c) (10 points) If $[0, 1] \supset I_1 \supset I_2 \supset \dots \supset I_n \supset \dots$ is a nested sequence of closed intervals, then $\bigcap_{n=1}^{\infty} I_n$ is non-empty.

(d) (10 points) For non-empty $A, B \subset \mathbb{R}$, let $A + B = \{x + y : x \in A, y \in B\}$. If A is open, then $A + B$ is open.

- (e) (10 points) Given sequences $\{x_n\}$ and $\{y_n\}$, define a new sequence $\{z_n\}$ by $z_{2n} = x_n$ and $z_{2n-1} = y_n$. The sequence $\{z_n\}$ converges if and only if $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n$ — that is, both sequences converge and have the same limit.

- (f) (10 points) If $f : D \rightarrow \mathbb{R}$ is a continuous function with domain $D \subset \mathbb{R}$, then for all $x_0 \in \bar{D}$, the closure of D , $\lim_{x \rightarrow x_0} f(x)$ exists.

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2. (20 points) Let $\{a_n\}$ be a Cauchy sequence, with $a_n \geq a > 0$. Working directly from the definitions, show that $\{a_n^{-2}\}$ is Cauchy.

3. (a) (5 points) Let $S = \{x \in \mathbb{R} : x^3 < x\}$. Determine $\sup S$ and $\inf S$.

(b) (15 points) Let $a_{n+1} = \frac{1}{2} \left(a_n + \frac{2}{a_n} \right)$, $a_1 = 1$. Set $A = \{x \in \mathbb{R} : x = a_n, n \in \mathbb{N}\} \subset \mathbb{R}$. Determine, $\limsup_{n \rightarrow \infty} a_n$, $\liminf_{n \rightarrow \infty} a_n$, $\inf A$, $\sup A$ and all limit points (if any) of A . (Hint: Show that, for $n \geq 1$, $2 \leq a_{n+1}^2$.)

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