Math 405 Midterm Exam 2 — Apr. 9, 2014

Name: ______________________________

- Complete the following problems. In order to receive full credit, please make sure to *justify your answers*. You are free to use results from class or the course textbook as long as you clearly state what you are citing.
- **You have 75 minutes.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted (nor are they needed). If you finish early, you must hand your exam paper to a proctor.
- Please check that your copy of this exam contains 6 numbered pages and is correctly stapled.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

The following boxes are strictly for grading purposes. Please do not mark.

Question	Points	Score
1	60	
2	15	
3	25	
Total	100	

1. Determine whether the following statements are true or false. Justify your answer (i.e., prove the claim, derive a contradiction or give a counter-example).

   (a) (10 points) If $f : D \to \mathbb{R}$ is continuous and D is closed, then for all $C \subset \mathbb{R}$ closed, $f^{-1}(C)$ is closed.

   (b) (10 points) If $f : D \to \mathbb{R}$ is continuous, and $D \subset \mathbb{R}$ is closed, then $f(D)$ is closed.

(c) (10 points) If $f : (a, b) \to \mathbb{R}$ is C^1 and injective, then $f' \neq 0$.

(d) (10 points) There is no differentiable function $f : (-1, 1) \to \mathbb{R}$ with $f'(x) = \left\{ \begin{array}{cc} -1 & x \leq 0 \\ 1 & x > 0 \end{array} \right.$.

(e) (10 points) Suppose $f : (-1, 1) \to \mathbb{R}$ is C^1 and $f(0) = 0$. If $f' = O(|x|), x \to 0$, then $f = O(|x|^2), x \to 0$.

(f) (10 points) If $f : (-1, 1) \to \mathbb{R}$ is C^3 and has Taylor polynomial at $x_0 = 0$ given by $T_3(f, 0; x) = 3 + x^2 - 100x^3$, then f has a strict local minimum at $x_0 = 0$.

2. (15 points) Let $f : (a, b) \to \mathbb{R}$ be uniformly continuous. Show that $\lim_{x \to b} f(x)$ exists.

3. (a) (5 points) Show that for any pairs (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$, there is a unique affine function g with $g(x_i) = y_i$, $i = 1, 2$.

   (b) (10 points) Let $f : (a, b) \to \mathbb{R}$ be C^2 and suppose that $f''(x) < 0$ for all $x \in (a, b)$. Show that if g is an affine function with $g(x_1) = f(x_1)$ and $g(x_2) = f(x_2)$, for $a < x_1 < x_2 < b$, then $g(x) < f(x)$ for all $x \in (x_1, x_2)$.

(c) (10 points) Let $f : (a,b) \to \mathbb{R}$ be C^2. Show that if, for all $a < y < z < b$, $f\left(\frac{y+z}{2}\right) \leq \frac{f(y)+f(z)}{2}$, then $f''(x) \geq 0$ for all $x \in (a,b)$. (Hint: If g is affine, then $g\left(\frac{y+z}{2}\right) = \frac{g(y)+g(z)}{2}$).

(This page intentionally left blank.)