Mathematic 405, Spring 2015: Assignment #2

Due: **Wednesday, February 11th**

Instructions: Please ensure that your answers are legible. Also make sure that sufficient steps are shown. Page numbers refer to the course text.

Problem #1. Consider the following relation on $X = \mathbb{Z} \times \mathbb{N}$

$$(x, y) \sim (x', y') \iff xy' = x'y.$$

a) Verify that $\sim$ is an equivalence relation on X. Let $Q = X/\sim$ be the quotient set.

b) Show that the map $\phi : X \to \mathbb{Q}$ given by $(x, y) \mapsto x/y$ is constant on each $\sim$-equivalence class $[(x, y)]$. Conclude that there is a well-defined map $\Phi : Q \to \mathbb{Q}$ and show that Φ is a bijection.

c) Define operations $\oplus$ and $\otimes$ on X by

$$(x, y) \oplus (x', y') = (xy' + yx', yy') \text{ and } (x, y) \otimes (x', y') = (xx', yy').$$

Verify that if $(x_1, y_1) \sim (x_2, y_2)$ and $(x_1', y_1') \sim (x_2', y_2')$, then $(x_1, y_1) \oplus (x_1', y_1') \sim (x_2, y_2) \oplus (x_2', y_2')$ and $(x_1, y_1) \otimes (x_1', y_1') \sim (x_2, y_2) \otimes (x_2', y_2')$. Conclude that there are well defined operations $\oplus$ and $\otimes$ on Q given by

$$[(x, y)] \oplus [(x', y')] = [(x, y) \oplus (x', y')] \text{ and } [(x, y)] \otimes [(x', y')] = [(x, y) \otimes (x', y')].$$

d) Show that $\Phi(z \oplus z') = \Phi(z) + \Phi(z')$ and $\Phi(z \otimes z') = \Phi(z)\Phi(z')$. That is, we have constructed a model of the rationals out of the integers.

Problem #2. p. 37 # 4

Problem #3. p. 37 # 5

Problem #4. p. 37 # 7

Problem #5. p. 48 # 2

Problem #6. p. 48 # 3

Problem #7. p. 48 # 10

Problem #8. The so-called Babylonian method for finding the square root of $S \in \mathbb{Q}^+$ is a recursive algorithm defined as follows: Start with any $x_1 \in \mathbb{Q}^+$ and define (for $n \geq 1$)

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{S}{x_n}\right).$$

a) Show $S \leq x_{n+1}^2$ and hence $x_{n+1} \leq x_n$ for $n \geq 1$. Conclude that the closed intervals $I_n = \left[\frac{S}{x_n}, x_n\right]$ contain $\sqrt{S}$ in the sense that $\left(\frac{S}{x_n}\right)^2 \leq S$ and $S \leq x_n^2$ and that $I_{n+1} \subset I_n$.

b) Show that the lengths of I_n satisfy $|I_{n+1}| \leq \frac{1}{2}|I_n|$.

c) Show that $\{x_n\}$ is a Cauchy sequence over the rationals.