Mathematic 405, Fall 2015: Assignment #5

Due: **Wednesday, March 11th**

Instructions: Please ensure that your answers are legible. Also make sure that sufficient steps are shown. Page numbers refer to the course text.

Problem #1. Use the intermediate value theorem to show that if $p(x) = \sum_{i=1}^{n} a_i x^i$ is a degree n polynomial (so $a_n \neq 0$) and n is odd, then p must have a real zero.

Problem #2. Let $f : [0,1] \to [0,1]$ be continuous. Use the intermediate value theorem to show that f has at least one fixed point – i.e., a point satisfying $f(x) = x$.

Problem #3. Show that if $f : (0,1) \to \mathbb{R}$ is uniformly continuous, then $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 1^-} f(x)$ both exist. Use this to show that there is a uniformly continuous function $\hat{f} : [0,1] \to \mathbb{R}$ with $\hat{f}(x) = f(x)$ for all $x \in (0,1)$. Give an example to show this is not possible if f is only continuous.

Problem #4. p. 125 # 6

Problem #5. p. 125 # 7

Problem #6. p. 138 # 10

Problem #7. p. 138 # 11

Problem #8. p. 138 # 12

Bonus Problem. (Will not be graded)

Let $C \subset \mathbb{R}$ be an arbitrary non-empty compact set and suppose $f : C \to C$ satisfies $|f(x) - f(y)| \leq \alpha |x - y|$ for all $x, y \in C$ and some $\alpha \in (0,1)$ (in particular f is Lipschitz). Such a map is an example of a *contraction.*

a) Show that f has a fixed point. (Hint: Show that the inductively defined sequence $a_1 = x_1, a_{n+1} = f(a_n)$ is Cauchy where here x_0 an arbitrary point of C).
b) Show that this fixed point is the only fixed point of f.
c) What happens if C is not compact or f is merely continuous.