Midterm 1 Solutions

1. (10 pts each) True or false; justify as much as you can.

a. The set S of all sequences consisting of zeroes and ones is countable.

False by Cantor's diagonalization argument. If the set (say S) was countable, i.e $S = \{b^1, b^2, \ldots, b^n, \ldots\}$ then define a new sequence $\{x_n\}$ with $x_n = 0$ if $b^n_n = 1$ and $x_n = 1$ otherwise. Then $\{x_n\}$ is not in the list. Alternatively define a map $f : 2^{\mathbb{N}} \to S$ by $f(A) = \{x_n\}$ where $x_n = 1$ if $n \in A$ and otherwise. It is easy to see that f is a bijection.

b. A sequence is convergent if and only if all of its subsequences are convergent.

True. It sounds false because at first glance you may have two subsequences $\{x_n'\}$ and $\{y_n'\}$ which have different limits. However this cannot happen because we can intersperse these subsequences (with ordering as they appear in the original sequence) and obtain a new subsequence $\{z_n'\}$ which does not converge.

c. $(n + 1)! \geq 2^n$ for all $n \in \mathbb{N}$.

True by induction: Let S be the set all $n \in \mathbb{N}$ for which this is true. Then S contains 1 and assuming S contains n, we have

$$(n + 2)! = (n + 2)(n + 1)! \geq (n + 2)2^n \geq 22^n = 2^{n+1}$$

so S contains $n + 1$. Hence $S = \mathbb{N}$.

d. The sup of a bounded infinite set S is the largest limit point of S.

This is false as stated since $\sup S$ might be an isolated point of S. If $\sup S$ is a limit point as well as an upper bound for S, it follows that any other limit point y of S must satisfy $y \leq \sup S$ for if $x_n \to y$, $x_n \in S$ then $x_n \leq \sup S$ so $y \leq \sup S$.

e. The subset $(-1, 1) \setminus \{0\}$ of $\mathbb{R}$ is open.

True. $(-1, 1) \setminus \{0\} = (-1, 0) \cup (0, 1)$ is the union of two open intervals so is open.

f. The countable union of closed intervals is closed.

False. Take $I_n = [0, 1 - \frac{1}{n}]$, $n = 2, 3, \ldots$. Then $\cup I_n = [0, 1)$ does not contain 1 .

2. (20 pts) Let $\{a_n\}$ be a Cauchy sequence. Show directly using the definition that the sequence $\{a_n^2\}$ is also a Cauchy sequence. Carefully justify all of the steps. You may use the result that a Cauchy sequence is bounded.

Proof. Since $\{a_n\}$ is Cauchy, $|a_n| \leq M$ for some $M > 0$. Then $|a_j^2 - a_k^2| = |(a_j - a_k)(a_j + a_k)| \leq 2M|a_j - a_k|$. Given $\varepsilon > 0$ choose $N = N(\varepsilon)$ so that $|a_j - a_k| \leq \frac{\varepsilon}{2M}$ for $j, k \geq N$. Then $|a_j^2 - a_k^2| \leq \varepsilon$ for $j, k \geq N$.

3. (20pts) Let $S = (-\infty, -1] \cup (1, 2) \cup \{3\}$. Find (5pts each)

a. The limit points of S.

The limit points are $(-\infty, -1] \cup [1, 2]$.

b. ∂S.

$\partial S = \{\text{closure of S}\} \setminus \{\text{interior of S}\} = \{-\infty\} \cup \{-1, 1, 2, 3\}$. Recall closure of S $= S \cup \{\text{limit points of S}\}$

c. The isolated points of S.

The point $\{3\}$ is isolated.

d. The complement of S in $\mathbb{R}$ ($S' = \mathbb{R} \setminus S$).

$S' = (-1, 1] \cup [2, 3) \cup (3, \infty)$.