Solutions Exam 1 — Oct. 7, 2017

1. Consider the following four functions. Determine which extend (i.e., can be analytically continued) to an entire function (i.e., a holomorphic function on the entire complex plane) and which cannot. Remember to justify your answers.

   (a) (10 points) $f_1(x+iy) = u(x,y) + iv(x,y)$ where $u(x,y) = x^2 + y^2 - 1$ and $v(x,y) = 2xy$. Hint: Consider the Cauchy-Riemann equations.

   Observe first that f_1 is defined everywhere in $\mathbb{C}$. If it were entire, then it would have to satisfy the Cauchy-Riemann equations. That is,

   $$\frac{\partial}{\partial \bar{z}} f_1 = 0 \iff \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

   We have for the given function that $\frac{\partial u}{\partial x} = 2x = \frac{\partial u}{\partial x}$ so the first equation is satisfied. However, $\frac{\partial u}{\partial y} = 2y \neq -2x = -\frac{\partial v}{\partial x}$. As such, f_1 cannot be holomorphic at any point $z \neq 0$ and so is not entire.

   (b) (10 points) $f_2(z) = z^2 - 1$.

   We have that for every z, that

   $$f'(z) = \lim_{h \to 0} \frac{f_2(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{(z+h)^2 - z^2}{h} = \lim_{h \to 0} \frac{2zh + h^2}{h} = \lim_{h \to 0}(2z + h) = 2z$$

   Hence, the complex derivative exists at every point $z \in \mathbb{C}$. That is, the function is holomorphic at every point and so is entire.

(c) (10 points) $f_3(z) = \sum_{n=1}^{\infty} n^{-n}(z-2)^n$

This power series has coefficients $a_n = n^{-n}$. We compute that $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} n^{-1} = 0$. Hence, by Hadamard's formula, the radius of convergence of this series is $R = \infty$. It follows that this function is holomorphic in the entire complex plane and hence is entire. This is because power series are holomorphic on their entire disks of convergence.

(d) (10 points) $f_4(z) = \sum_{n=1}^{\infty} n^2 z^n$.

This power series has coefficients $b_n = n^2$. We compute that $\lim_{n\to\infty} |b_n|^{1/n} = \lim_{n\to\infty} n^{2/n} = 1$. As such the power series has radius of convergence 1. This means that f_4 is holomorphic in the disk $|z| < 1$. However, as the power series diverges for $|z| > 1$, the function f_4 cannot be extended to an entire function. This is because if there was a a F_4 that was entire and that extended f_4, then the Cauchy inequalities would imply that there was would be a power series $\sum_{n=0}^{\infty} c_n z^n$ with infinity radius of convergence so that $F_4(z) = \sum_{n=0}^{\infty} c_n z^n$. As this agrees with f_4 on $|z| < 1$, this means that $c_n = b_n$. This yields a contradiction with what we already computed and so we conclude that there can be no such F_4.

2. (a) (20 points) Use a contour integral to carry out the the following computation:

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1} dx = \frac{\pi}{e}.$$

Hint: Use as contours the semi-circles of radius R with keyhole at $z = i$ and that if $z = x + iy$ and $y \geq 0$, then $|\exp(iz)| = \exp(-y) \leq 1$.

Following the hint, we consider the semicircle made up of L_R which is the segment $[-R, R]$ on the real axis and C_R^+, the upper half of ∂D_R. We turn this into a keyhole, joining the small circle $C_\epsilon = \partial D_\epsilon(i)$ by two segments parallel to the imaginary axis connecting the small circle to L_R. Denote this keyhole contour by $\Gamma_{\delta,\epsilon,R}$ (I suggest you draw this to understand the argument).

Let $f(z) = \frac{e^{iz}}{z^2+1}$ we observe that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1} dx = \lim_{R\to\infty} \mathrm{Re} \int_{L_R} f(z)dz.$$

As f is holomorphic when $z \neq \pm i$, i.e., in the interior of $\Gamma_{\delta,\epsilon,R}$, and so by Cauchy's theorem,

$$0 = \int_{\Gamma_{\delta,\epsilon,R}} f(z)dz.$$

As the two line segments cancel out when $\delta \to 0$, by taking $\delta \to 0$ we obtain

$$0 = \lim_{\delta\to 0} \int_{\Gamma_{\delta,\epsilon,R}} f(z)dz = \int_{L_R} f(z)dz + \int_{C_R^+} f(z)dz + \int_{C_\epsilon} f(z)dz.$$

Observe that, when $R > 4$

$$\left| \int_{C_R^+} f(z)dz \right| \leq \sup_{C_R^+} |f(z)| Len(C_R^+) \leq \frac{2\pi R}{R^2} = \frac{2\pi}{R}$$

where here we used the hint and the fact that on C_R^+ one has $z = x + iy$ with $y \geq 0$. Indeed,

$$|f(z)| = \left| \frac{e^{iz}}{z^2+1} \right| = \frac{|e^{iz}|}{|z^2+1|} \leq \frac{2}{R^2}$$

where we used that when $R > 4$

$$|z^2+1| = |x+1+iy|^2 = (x+1)^2+y^2 = R^2+2x+1 = \frac{1}{2}R^2+\frac{1}{2}R^2+2x+1 > \frac{1}{2}R^2+2R+2x+1 > \frac{1}{2}R^2.$$

One checks (using a power series for instance) that near $z = i$,

$$f(z) = -\frac{1}{2ei(z-i)} + F(z)$$

where $F(z)$ is holomorphic near $z = i$. Hence, by Cauchy's theorem when ϵ is small,

$$\int_{C_\epsilon} f(z)dz = \int_{C_\epsilon} -\frac{1}{2ei(z-i)} dz = -\frac{1}{2ei} \int_{C_\epsilon} \frac{1}{z-)} dz = -\frac{\pi}{e}.$$

Thus,

$$\int_{L_R} f(z)dz = \frac{\pi}{e} + O(R^{-1}).$$

Verifying the claim.

3. (a) (10 points) Compute $\int_{\partial D_1(0)} \bar{z} dz$ where $\partial D_1(0)$ is the unit circle with positive orientation. Use this to explain why there is no holomorphic function $f : D_2(0) \to \mathbb{C}$ with $f(z) = \bar{z}$ for $z \in \partial D_1(0)$.

Using the parametrization $z(t) = e^{it}$ we compute directly that

$$\int_{\partial D_1(0)} \bar{z} dz = \int_0^{2\pi} \bar{z}(t) z'(t) dt = \int_0^{2\pi} e^{-it} i e^{it} dt = i \int_0^{2\pi} dt = 2\pi i.$$

As $\bar{D}_1(0) \subset D_2(0)$, it there was such a holomorphic f, then Cauchy's theorem would give

$$2\pi i = \int_{\partial D_1(0)} \bar{z} dz = \int_{\partial D_1(0)} f(z) dz = 0$$

which is clearly absurd and means there can be no such f.

(b) (10 points) Show that if $g : D_2(0)\setminus\{0\} \to \mathbb{C}$ is holomorphic and $g(z) = \bar{z}$ for $z \in \partial D_1(0)$, then $g(z) = \frac{1}{z}$. Why does this not contradict part a)?

We observe that that $z \in \partial D_1(0)$ if and only if $1 = |z|^2 = z\bar{z}$. As $g(z) = 1/z$ is holomorphic on $D_2(0)\setminus\{0\}$ and has the desired behavior of $\partial D_1(0)$. If $h(z)$ was some other function holomorphic on $D_2(0)\setminus\{0\}$ with $h(z) = \bar{z}$ on $\partial D_1(0)$, then $H(z) = g(z) - h(z)$ is holomorphic on $D_2(0)\setminus\{0\}$ and vanishes on $\partial D_1(0)$. Notice any point in $\partial D_1(0)$ is an accumulation points of the zeros of H and so, as $D_2(0)\setminus\{0\}$ is connected, $H(z)$ vanishes identically.

Finally, this does not contradict part a) as $\bar{D}_1(0)$ is not a subset of $D_2(0)\setminus\{0\}$ and so Cauchy's theorem does not apply.

4. (20 points) Use the Cauchy inequalities to show that if f is an entire function that satisfies

$$|f(z)| \leq C|z|\log(1+|z|),$$

for all $z \in \mathbb{C}$, then $f(z) = 0$ for all $z \in \mathbb{C}$. Hint: Determine $\lim_{r\to\infty} \frac{\log(1+r)}{r}$ and $\lim_{r\to 0^+} \frac{\log(1+r)}{r}$ and use the first limit to show $f(z) = az + b$.

We first observe that by L'Hopital's rule,

$$\lim_{r\to\infty} \frac{\log(1+r)}{r} = \lim_{r\to\infty} \frac{1}{r+1} = 0.$$

Similarly, as $\log(1) = 0$, we can use L'Hopital's rule, to see that

$$\lim_{r\to 0^+} \frac{\log(1+r)}{r} = \lim_{r\to 0^+} \frac{1}{r+1} = 1.$$

As f is entire, it follows from the Cauchy inequalities, that for all $r > 0$

$$|f^{(n)}(0)| \leq \frac{n!}{r^n} \sup_{z\in\partial D_r(0)} |f(z)|$$

Using the estimate,

$$|f^{(n)}(0)| \leq \frac{Cn!}{r^{n-1}} \log(r+1).$$

When $n \geq 2$ this yields

$$|f^{(n)}(0)| \leq \frac{Cn!}{r^{n-2}} \frac{\log(r+1)}{r}.$$

As $\frac{Cn!}{r^{n-2}}$ is bounded as $r \to \infty$ and $\frac{\log(r+1)}{r} \to 0$ as $r \to \infty$ we conclude that $f^{(n)}(0) = 0$ for $n \geq 2$. Hence, $f(z) = a + bz$ (to see this consider the power series of f centered at $z = 0$.)

However, $|f(0)| \leq |0|$ so $a = 0$ and

$$|f'(0)| = \left|\lim_{h\to 0} \frac{f(h) - f(0)}{h}\right| = \lim_{h\to 0} \frac{|f(h)|}{|h|}.$$

Hence, using the estimate

$$|f'(0)| \leq \lim_{h\to 0} \frac{C|h|\log(|h|+1)}{|h|} = \lim_{r\to 0^+} \frac{Cr\log(|r|+1)}{r} = C \lim_{r\to 0^+} r \lim_{r\to 0^+} \frac{\log(|r|+1)}{r} = 0.$$

Hence, $f'(0) = b = 0$ and so $f(z) = 0$ for all z.