Solutions Exam 2 — Dec. 7, 2017

1. Suppose that $f : D_1^*(0) \to \mathbb{C}$ is a holomorphic function satisfying $|f(z)| \leq C|z|^{-3/2}$. Here $D_1^*(z) = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and $C > 1$.

   (a) (15 points) Show that f has either a simple pole at $z = 0$ or a removable singularity. Hint: Consider $h(z) = z^2 f(z)$.

   Let $h(z) = z^2 f(z)$. We have $|h(z)| = |z^2||f(z)| \leq C|z|^{1/2} \leq C$ on D_1^*. This means h is bounded and holomorphic in D_1^*. Hence, by the Riemann removable singularities theorem, 0 is a removable singularity for h and so h extends to a holomorphic function $H(z)$ on all of D_1. Notice, that $|H(0)| = \lim_{z\to 0} |H(z)| \leq \lim_{z\to 0} C|z|^{1/2} = 0$ and so H has a zero at $z = 0$. In particular, we can write $H(z) = zG(z)$ where G is holomorphic in D_1. Clearly, $f(z) = z^{-1}G(z)$ and so either f has a simple pole at $z = 0$ (if $G(0) \neq 0$) or f has a removable singularity at $z = 0$ (if $G(0) = 0$).

   (b) (5 points) Show that if, in addition, $|f(z)| \geq |z|^{-1/2}$, then f has a simple pole at $z = 0$.

   The lower bound ensures that $\lim_{z\to 0} |f(z)| \geq \lim_{z\to 0} |z|^{-1/2} = \infty$ and so f has a pole at $z = 0$. By the previous part this pole is a simple pole.

2. Compute the following contour integrals.

(a) (10 points) $\int_{\partial D_1(0)} \frac{e^{z^2}\cos(z)}{z^2} dz$

As the integrand is meromorphic in D_1, the residue theorem gives

$$\int_{\partial D_1(0)} \frac{e^{z^2}\cos(z)}{z^2} dz = 2\pi i \mathrm{Res}_{z=0} \frac{e^{z^2}\cos(z)}{z^2} = 0.$$

In order to get the last equality we note that, $\frac{e^{z^2}\cos(z)}{z^2}$ has a pole of order 2 at $z = 0$. Hence, in order to compute the residue we use the first few terms in the Taylor expansion

$$e^{z^2} = 1 + z^2 + O(z^4)$$

and

$$\cos(z) = 1 + \frac{z^2}{2} + O(z^4)$$

to see that

$$\frac{e^{z^2}\cos(z)}{z^2} = \frac{1}{z^2} + \frac{3}{2} + O(z^2)$$

and so the residue at $z = 0$ is zero.

One could also use the residue formula for higher order poles

$$\mathrm{Res}_{z=0} \frac{e^{z^2}\cos(z)}{z^2} = \frac{1}{1!} \lim_{z\to 0} \frac{d}{dz} z^2 \left(\frac{e^{z^2}\cos(z)}{z^2} \right) = \lim_{z\to 0} \left(2ze^{z^2} - \sin(z) \right) = 0.$$

(b) (10 points) Let f be holomorphic on $D_3(0)$ and suppose $f(1) = f'(1) = -1$, $\int_{\partial D_2(0)} \frac{f(\zeta)}{(\zeta-1)^2} d\zeta$.

As f is holomorphic in D_3, the generalized Cauchy integral formula gives

$$-1 = f'(1) = \frac{1!}{2\pi i} \int_{\partial D_2(0)} \frac{f(\zeta)}{(\zeta-1)^2} d\zeta.$$

Hence,

$$\int_{\partial D_2(0)} \frac{f(\zeta)}{(\zeta-1)^2} d\zeta = -2\pi i.$$

One could also use the residue theorem.

(c) (10 points) Let f be holomorphic on $D_3(0)$ and $f(1) = 2$ and $f(-1) = 1$. Compute $\int_{\partial D_2(0)} \frac{f(z)}{z^2-1} dz$.

We note that $z^2 - 1 = (z-1)(z+1)$ and so $\frac{f(z)}{z^2-1}$ has simple poles at $z = 1$ and $z = -1$. Hence, the residue theorem implies that

$$\int_{\partial D_2(0)} \frac{f(z)}{z^2-1} dz = 2\pi i \mathrm{Res}_{z=1} \frac{f(z)}{z^2-1} + 2\pi i \mathrm{Res}_{z=-1} \frac{f(z)}{z^2-1}.$$

As the poles are simple we have

$$\mathrm{Res}_{z=1} \frac{f(z)}{z^2-1} = \lim_{z\to 1}(z-1)\frac{f(z)}{z^2-1} = \lim_{z\to 1} \frac{f(z)}{z+1} = \frac{f(1)}{1+1} = 1$$

and

$$\mathrm{Res}_{z=-1} \frac{f(z)}{z^2-1} = \lim_{z\to -1}(z+1)\frac{f(z)}{z^2-1} = \lim_{z\to -1} \frac{f(z)}{z-1} = \frac{f(-1)}{-1-1} = -\frac{1}{2}.$$

Hence,

$$\int_{\partial D_2(0)} \frac{f(z)}{z^2-1} dz = \pi i.$$

3. Explain why there is no holomorphic function with the given domain and properties.

(a) (10 points) A $f : \mathbb{C} \to \mathbb{C}$ with $f(5) = 0$ and $f\left(\frac{1}{n}\right) = 1$ for all $n \in \mathbb{Z}$, $n \geq 1$. Hint: what is happening at $z = 0$.

We observe that $\lim_{n\to\infty} \frac{1}{n} = 0$ and hence 0 is a point of accumulation of the sequence $\left\{\frac{1}{n}\right\}_{n\geq 1}$. This means that f must identically be equal to 1 by the analytic continuation property of holomorphic functions. This is inconsistent with $f(5) = 0$ and so there can be no such holomorphic f.

(b) (10 points) A $f : D_2(0) \to \mathbb{C}$ with $f(0) = -2$ and $|f(z)| \leq 1$ on $\partial D_1(0)$

As f is holomorphic on D_2 it is continuous on $\bar{D}_1$. Hence, by the maximum modulus principle one should have

$$2 = |f(0)| \leq \max_{\partial D_1} |f(z)| \leq 1.$$

As this is absurd, there can be no such holomorphic f. One could also see this using the Cauchy integral formula.

(c) (10 points) A $f : D_1^*(0) \to \mathbb{C}$ with $1 \leq |f(z)|$ for all $z \in D_1^*(0)$ and so that $\lim_{z\to 0} |f(z)|$ does not exist. Here $D_1^*(0) = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

By hypotheses, $f(z)$ never vanishes on D_1^* and so $g(z) = \frac{1}{f(z)}$ is holomorphic on D_1^*. Notice that

$$|g(z)| = \frac{1}{|f(z)|} \leq 1$$

Hence, g is bounded and so, by the Riemann removable singularity theorem, g extends to a holomorphic function of D_1. As such $\lim_{z\to 0} |g(z)| = |g(0)|$ exists. This means $\lim_{z\to 0} |f(z)|$ also exists (though could be ∞ if $g(0) = 0$) and so there is no such f.

Alternatively, one could use that $\lim_{z\to 0} |f(z)|$ does not exist only when f has an essential singularity at $z = 0$. The hypotheses that $1 \leq |f(z)|$ contradicts the Casorati-Weierstrass theorem and so shows that there can be no such f.

4. Show (by construction) that there is a holomorphic function f with the given properties.

(a) (10 points) A simply connected domain Ω and an $f : \Omega \to \mathbb{C}$ so that for all $z \in \Omega$, $(f(z))^2 = z$ and $f(1) = 1$ while $f(4) = -2$. Hint: Draw the right domain and use the corresponding logarithm.

Pick a simply connected domain Ω with the property that $1 \in \Omega$, $4 \in \Omega$, $0 \notin \Omega$ and Ω "winds" clockwise one around 0. (e.g., consider a small neighborhood of the curve $t \mapsto (t+1)^2 e^{2\pi i t}$, $t \in [0,1]$). For such Ω, $\log_\Omega(1) = 0$, while $\log_\Omega(4) = \log 4 + 2\pi i = 2\log 2 + 2\pi i$.
Letting $f(z) = e^{\frac{1}{2}\log_\Omega(z)}$ one has that f is holomorphic on Ω and

$$(f(z))^2 = \left(e^{\frac{1}{2}\log_\Omega(z)}\right)^2 = e^{\log_\Omega(z)} = z.$$

Moreover, $f(1) = e^{\frac{1}{2}\log_\Omega(1)} = e^0 = 1$ while $f(4) = e^{\frac{1}{2}\log_\Omega(4)} = e^{\log 2 + \pi i} = -2$. Verifying the claim.

(b) (10 points) A $f : \mathbb{C} \to \mathbb{C}$ with simple zeros at $z = 2^n$ for all $n \in \mathbb{Z}$, $n \geq 0$.

Consider the infinite produce

$$f(z) = \Pi_{n=0}^{\infty} \left(1 - \frac{z}{2^n}\right).$$

We note that this product converges uniformly on any D_R as on this disk

$$\sum_{n=0}^{\infty} \left|\frac{z}{2^n}\right| = |z| \sum_{n=0}^{\infty} \frac{1}{2^n} \leq R \sum_{n=0}^{\infty} \frac{1}{2^n} < \infty.$$

Moreover, as this sum is finite this product only has simple zeros at $z = 2^n$ when $n = 0, 1, \ldots, \infty$. As such the product gives an entire function with simple zeros *only* at $z = 2^n$
Alternatively, one can observe that a function that has simple zeros at (say) every integer numbers also has simple zeros at each 2^n. For instance, $f(z) = \sin(\pi z)$ is an example.