Solutions Exam 2 — May 2, 2018

1. (a) (10 points) State the definition of a measurable function $f : E \to \mathbb{R}$.

A function $f : E \to \mathbb{R}$ is measurable if and only if E is a measurable subset and for each $\alpha \in \mathbb{R}$, $f^{-1}((-\infty, \alpha)) = \{x \in E : f(x) < \alpha\}$ is a measurable set.

(b) (15 points) Let $f : \mathbb{R} \to \mathbb{R}$ be measurable and $g : \mathbb{R} \to \mathbb{R}$ be continuous. Prove that $g \circ f$ is measurable.

As g is continuous $U_\alpha = g^{-1}((-\infty, \alpha)) = U_\alpha$ is an open set. A basic property of a measurable function f is that for any open set U, $f^{-1}(U)$ is measurable, hence $(g \circ f)^{-1}((-\infty, \alpha)) = f^{-1}(U_\alpha)$ is measurable and so $g \circ f$ is also measurable. To see why this basic property holds, we observe that for any open interval, I, $f^{-1}(I)$ is measurable. Indeed, if $I = (a, b)$ and $b < \infty$, then $f^{-1}(I) \in \mathcal{M}$ by definition when $a = -\infty$ and when $a \in \mathbb{R}$

$$f^{-1}(I) = f^{-1}((-\infty, b)) \setminus \bigcup_{n=1}^{\infty} f^{-1}((-\infty, a - \frac{1}{n}))$$

and so the measurablity of $f^{-1}(I)$ follows from the definition and the σ-algebra property of $\mathcal{M}$. When $b = \infty$ one has

$$f^{-1}(I) = \bigcup_{n=1}^{\infty} f^{-1}(I \cap (-\infty, n))$$

and each $f^{-1}(I \cap (-\infty, n)) \in \mathcal{M}$ by the previous observation and so $f^{-1}(I) \in \mathcal{M}$ by σ-algebra property. As any open set, U, is the countable union of open intervals, i.e. $U = \cup_{n=1}^{\infty} I_n$ for open intervals I_n. The fact that $f^{-1}(U) \in \mathcal{M}$ follows from the σ-algebra property of $\mathcal{M}$.

2. Let $f : \mathbb{R} \to [0, \infty]$ be a non-negative measurable function.

   (a) (10 points) State the definition of the Lebesgue integral of f.

Given f, the Lebesgue integral of f is defined to by

$$\int f = \sup \left\{ \int \phi : 0 \leq \phi \leq f, \text{ and } \phi \text{ is a simple function} \right\}.$$

Here ϕ is simple means that $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$ where $a_i \in \mathbb{R}$ and $E_i \in \mathcal{M}$ and

$$\int \phi = \int \sum_{i=1}^{n} a_i \chi_{E_i} = \sum_{i=1}^{n} a_i m(E_i).$$

   (b) (10 points) Directly using the definition, show that

$$f(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ \infty & x = 0, \end{cases}$$

is *not* Lebesgue integrable.

Recall, f is Lebesgue integrable if and only if $\int f < \infty$. Consider the sequence of simple functions $\phi_n = n^2 \chi_{(0, \frac{1}{n})}$. Clearly, $0 \leq \phi_n \leq f$ and one has

$$\int \phi_n = n^2 m((0, \frac{1}{n})) = n$$

Hence, $\int f \geq n$ for all $n \geq 1$ and so $\int f = \infty$. That is, f is not Lebesgue integrable.

(c) (10 points) Show that $f(x) = \frac{1}{1+x^2}$ is Lebesgue integrable, i.e., is in $L_1(\mathbb{R})$.

First observe that f is continuous and hence measurable. Let $f_n(x) = \chi_{[-n,n]} f(x)$. Clearly each f_n is measurable and the sequence (f_n) is monotone and $\lim_{n\to\infty} f_n = f$ pointwise. Hence, by the monotone convergence theorem

$$\int f = \lim_{n\to\infty} \int f_n = \lim_{n\to\infty} \int_{-n}^{n} \frac{1}{1+x^2}.$$

As $\frac{1}{1+x^2}$ is continuous and $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$ one can use the fundamental theorem of calculus to see that

$$\int_{-n}^{n} \frac{1}{1+x^2} = \arctan(n) - \arctan(-n).$$

Hence,

$$\int f = \lim_{n\to\infty} \int_{-n}^{n} \frac{1}{1+x^2} = \lim_{n\to\infty} (\arctan(n) - \arctan(-n)) = \pi < \infty$$

and so f is Lebesgue integrable.

3. Let (f_n) be a sequence of Lebesgue integrable functions (i.e., $f_n \in L_1(\mathbb{R})$) that converge pointwise to the zero function.

   (a) (10 points) Show, by example, that it is possible for the f_n to *not* converge in $L_1(\mathbb{R})$.

   Let $f_n(x) = \chi_{[n,n+1]}$. For each x, as long as $n \geq |x| + 1$, one has $f_n(x) = 0$. Hence, $\lim_{n\to\infty} f_n = 0$ pointwise. However, $\int |f_n - f_{n+2}| = \int f_n + f_{n+2} = 2$ and so the sequence (f_n) is not Cauchy in $L_1(\mathbb{R})$ and hence cannot converge to the zero function or to any other function.

   (b) (10 points) Suppose that, in addition, the f_n satisfy $|f_n(x)| \leq \frac{1}{1+x^2}$ for a.e. $x \in \mathbb{R}$. Show that the f_n converge in $L_1(\mathbb{R})$ to the zero function.

   In a previous problem we established that $g(x) = \frac{1}{1+x^2}$ is Lebesgue integrable. Hence, by the Lebesgue dominated convergence theorem one has

$$\lim_{n\to\infty} \int |f_n| = \int \lim_{n\to\infty} |f_n| = \int 0 = 0.$$

   That is, f_n converges to 0 in $L_1(\mathbb{R})$.

4. (a) (15 points) Show that there is a continuous function $g : [-\pi, \pi] \to \mathbb{R}$ with Fourier series given by

$$\sum_{k=1}^{\infty} 2^{-k} \cos kx.$$

What is the value of $\|g\|_2^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} g^2(x)\, dx$?

Notice that on $[-\pi, \pi]$ one has,

$$\sup_{[-\pi,\pi]} |2^{-k} \cos kx| \leq 2^{-k}$$

and the series $\sum_{k=1}^{\infty} 2^{-k} = 1 < \infty$ is summable. Hence, by the Weierstrass M-test, the functions

$$s_n(x) = \sum_{k=1}^{n} 2^{-k} \cos kx \in \mathcal{T}_n$$

are continuous and satisfy $s_n \to g$ uniformly for some continuous function $g : [-\pi, \pi] \to \mathbb{R}$. We claim g has the desired Fourier series. To see this, we observe that for $g \geq k$

$$a_k(s_n) = \frac{1}{\pi} \int_{-\pi}^{\pi} s_n(x) \cos(kx) dx = 2^{-k}$$

Hence, for $k \geq 1$

$$a_k(g) = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos(kx) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (\lim_{n\to\infty} s_n(x)) \cos(kx) dx = \lim_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} s_n(x) \cos(kx) dx$$

where we are allowed to interchange limits and integration as the convergence is uniform. Hence,

$$a_k(g) = 2^{-k}.$$

Similarly, one sees $b_k(g) = 0$ for all k and $a_0(g) = 0$. That is, g has the desired Fourier series. Finally, by using Parseval's identity we can compute that

$$\|g\|_2^2 = \frac{a_0(g)^2}{2} + \sum_{k=1}^{\infty} \left(a_k(g)^2 + b_k(g)^2\right) = \sum_{k=1}^{\infty} (2^{-k})^2 = \sum_{k=1}^{\infty} 4^{-k} = \frac{1}{3}.$$

(b) (10 points) Show that there is no continuous function $f : [-\pi, \pi] \to \mathbb{R}$ with Fourier series

$$\sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k}} \cos kx + \frac{1}{k} \sin kx \right).$$

If there was such a f, then one would have

$$s_n(f) = \sum_{k=1}^{n} \left(\frac{1}{\sqrt{k}} \cos kx + \frac{1}{k} \sin kx \right)$$

And hence, by Bessel's inequality and the orthogonality of the trigonometric functions,

$$\sum_{k=1}^{n} \left(\frac{1}{k} + \frac{1}{k^2} \right) = \|s_n(f)\|_2^2 \leq \|f\|_2^2 < \infty.$$

Here the last inequality follows as f is continuous and hence bounded on $[-\pi, \pi]$. However,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

as the harmonic series diverges. This contradicts,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} \leq \sum_{k=1}^{n} \left(\frac{1}{k} + \frac{1}{k^2} \right) \leq \|f\|_2^2 < \infty.$$

That is, there can be no such f.