Math 645, Fall 2020: Assignment #1

Due: **Tuesday, September 15th**

Problem #1. Let M and N be differentiable manifolds, show that there is a differentiable manifold structure on the cartesian product $M \times N$ so that the two natural projection maps $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are smooth.

Problem #2. Let M and N be topological spaces and $\phi : M \to N$ be a homeomorphism. Show that if N is a differentiable manifold, then there is a smooth atlas on M so that ϕ is diffeomorphism.

Problem #3. Let $\mathbb{R}$ denote the differentiable manifold coming from the standard atlas on $\mathbb{R}$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be given by $\phi(x) = x^3$ and let $\mathbb{R}'$ denote the differentiable manifold (i.e., the structure on $\mathbb{R}$) that makes ϕ a diffeomorphism. Describe $C^\infty(\mathbb{R}') \cap C^\infty(\mathbb{R})$.

Problem #4. Let $X = \mathbb{R} \coprod \{0'\}$ be the disjoint union of of the real numbers with a distinct point $0'$. Let $U = X \setminus \{0'\}$ and $V = X \setminus \{0\}$ be two subsets on X. Let $\phi_U : U \to \mathbb{R}$ be the identity and let $\phi_V : V \to \mathbb{R}$ be identity away from $0'$ and with $\phi_V(0') = 0$. Show these two charts are smoothly compatible, but do not give rise to a smooth manifold structure on X. Hint Think about the Hausdorff propery.

Problem #5. Let $\mathbb{RP}^n$ be the space of lines in $\mathbb{R}^{n+1}$ that go through the origin. We identify this space with the quotient $(\mathbb{R}^{n+1} \backslash \{0\})/\sim$ where $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n) \iff (a_0, \ldots, a_n) = \lambda(b_0, \ldots, b_n)$ for some $\lambda \neq 0$ (more precisely, the identification is made by picking a point on the line other than the origin). Denote the equivalence class of $(a_0, \ldots, a_n)$ by $[a_1, \ldots, a_n]$. We introduce "charts" (U_i, ϕ_i) as follows:

$$U_i = \{[a_0, \ldots, a_n] : a_i \neq 0\} \text{ and } \phi_i([a_0, \ldots, a_n]) = \frac{1}{a_i}(a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$$

a) Show that one can topologize $\mathbb{RP}^n$ so that this (U_i, ϕ_i) forms a smooth atlas.
b) Consider the map $\pi : \mathbb{S}^n \to \mathbb{RP}^n$ given by sending a point on the sphere to the line through it and the origin. Show this map is smooth.

Problem #6. Let M be differentiable manifold and $X, Y \in \mathcal{X}_0(M)$ be two vector fields with compact support. If $\phi_t : M \to M$ is the time t flow of X and ψ_t is the time t flow of Y show that

$$\frac{d}{dt}|_{t=0}\gamma_t(p) = [X, Y](p).$$

where

$$\gamma_t = \psi_{-\sqrt{t}} \circ \phi_{-\sqrt{t}} \circ \psi_{\sqrt{t}} \circ \phi_{\sqrt{t}}.$$