Math 645, Fall 2020: Assignment #2

Due: **Tuesday, September 22nd**

Problem #1. Show that for vector fields $X, Y_1, \ldots, Y_k \in \mathcal{X}(M)$ and a $A \in \mathcal{T}^k(M))$,

$$(L_X A)(Y_1, \ldots, Y_k) = X \cdot (A(Y_1, \ldots, Y_n)) - \sum_{i=1}^{k} A(Y_1, \ldots, Y_{i-1}, [X, Y_i], Y_{i+1}, \ldots, Y_k).$$

Problem #2. Show that $\mathbb{RP}^2$ is non-orientable.

Problem #3. Let M be an n-dimensional differentiable manifold. Define the space of frames of M at p, F_pM, to be the set of ordered bases of T_pM – i.e.,

$$F_pM = \{(v_1, \ldots, v_n) : v_i \in T_pM \text{ and } \{v_1, \ldots, v_n\} \text{ is an ordered basis of } T_pM\}.$$

Let $FM = \bigcup_{p \in M} F_pM$ be the *frame bundle* of M. Define the following equivalence relation on FM, $(p, (v_1, \ldots, v_n)) \sim (q, (w_1, \ldots, w_n)) \iff p = q$ and $v_i = \sum_{j=1}^{n} A_{ij} w_j$ where $A_{ij} \in GL(n, \mathbb{R})$ has positive determinant. Let $\bar{M} = FM/\sim$ be the set of equivalence classes.

a) Prove that FM has a natural differentiable manifold structure and that the map $\Pi : FM \to M$ given by $(p, (v_1, \ldots, v_n)) \mapsto p$ is smooth.
b) Show that $\bar{M}$ has a natural differentiable manifold structure and that the map $\pi : \bar{M} \to M$ defined by $[p, (v_1, \ldots, v_n)] \mapsto p$ is a local diffeomorphism.
c) Show that $\bar{M}$ is an orientable n-dimensional manifold.
d) Show that $\bar{M}$ is connected if and only if M is connected and non-orientable. That is, $\bar{M}$ is the *oriented double cover* of M.

Problem #4.

a) Let M and N be differentiable manifolds. Prove that $M \times N$ is orientable if and only if both M and N are orientable.
b) Prove that TM is orientable (even if M is not).
c) Prove that $T\mathbb{RP}^2$ is not diffeomorphic to $\mathbb{RP}^2 \times \mathbb{R}^2$.

Problem #5. Show the following:

a) If U and V are open subsets of $\mathbb{R}^n$ and $\phi : U \to V$ a diffeomorphism, then when $q = \phi(p)$,

$$(\phi^* dx^1 \wedge \ldots \wedge dx^n)_p = (\det d\phi_p) dx^1_p \wedge \ldots \wedge dx^n_p.$$

b) If $U \subset \mathbb{R}^n$ is an open set, then

$$\Lambda^n U = \{f dx^1 \wedge \ldots \wedge dx^n : f \in C^\infty(U)\}.$$

That is, $\Lambda^n U$ is a the trivial rank one vector bundle.
c) For connected M, $\Lambda^n M$ is a rank one vector bundle that is trivial if and only if M is orientable.