Math 645, Fall 2020: Assignment #3

Due: **Tuesday, October 6th**

Problem #1. Let (M, g) be a Riemannian manifold with Riemannian distance d_g. Show that the manifold topology and the topology of the metric space (M, d_g) coincide.

Problem #2. Let (M, g) and (N, h) be Riemannian manifolds. Show that if $\phi : M \to N$ is a Riemannian isometry (i.e., it is a diffeomorphism and $\phi^* h = g$), then it is a metric isometry (i.e., $d_h(\phi(p), \phi(q)) = d_g(p, q)$.

Problem #3. Let $i : M \to \mathbb{R}^n$ be an immersion and suppose $g = i^* \bar{g}$ (so (M, g) is isometrically immersed in Euclidean space).

a) Show that $d_{\bar{g}}(i(p), i(q)) \leq d_g(p, q)$.
b) Show that if one has $d_g(p, q) = d_{\bar{g}}(i(p), i(q))$ for all pairs of $p, q \in M$, then (M, g) is flat. You may use without proof the fact that the shortest curve connecting two points in Euclidean space must be a (reparamterization) of a line segment.

Problem #4. Let $SL(2, \mathbb{R})$ denote the set of 2×2 matrices with determinant 1. Let $SL(2, \mathbb{R})$ act on $\mathbb{U}^2 = \{x + iy : y > 0\} \subset \mathbb{C}$, the upper half plane thought of as a subset of $\mathbb{C}$, by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}$$

Show that this gives an isometry on $(\mathbb{U}^2, \breve{g}_1)$, the upper half-plane model of hyperbolic space of radius 1. Show this action is transitive. Recall,

$$\breve{g}_1 = \frac{dx^2 + dy^2}{y^2}.$$

Problem #5. Define a connection ∇ on $\mathbb{R}^3$ by

$$\begin{aligned}
&\nabla_{\partial_{x_1}} \partial_{x_1} = \nabla_{\partial_{x_2}} \partial_{x_2} = \nabla_{\partial_{x_3}} \partial_{x_3} = 0, \\
&\nabla_{\partial_{x_1}} \partial_{x_2} = \partial_{x_3}, && \nabla_{\partial_{x_2}} \partial_{x_1} = -\partial_{x_3}, \\
&\nabla_{\partial_{x_1}} \partial_{x_3} = -\partial_{x_2}, && \nabla_{\partial_{x_3}} \partial_{x_1} = \partial_{x_2}, \\
&\nabla_{\partial_{x_2}} \partial_{x_3} = \partial_{x_1}, && \nabla_{\partial_{x_3}} \partial_{x_2} = -\partial_{x_1},
\end{aligned}$$

where here $\partial_{x_i} = \frac{\partial}{\partial x^i}$ are the coordinate vector fields of the standard coordinates.

a) Verify that this connection is compatible with $\bar{g}$ the Euclidean metric but is not torsion-free.
b) Determine the zero-acceleration curves of ∇.
c) Determine the parallel transport of ∇ along zero-acceleration curves.