Math 645, Fall 2020: Assignment #5

Due: **Tuesday, October 13th**

Problem #1. Let (M, g) be a Riemannain manifold with Levi-Civita connection ∇. Show that for any one form $\eta \in \Omega^1(M)$ and vectorfields $X, Y \in \mathfrak{X}(M)$ one has

$$d\eta(X, Y) = \nabla_X \eta(Y) - \nabla_Y \eta(X).$$

Problem #2. Let (M, g) be a closed Riemannian manifold. We say a vector field $X \in \mathcal{X}(M)$ is a *Killing field*, provided $\phi_t^* g = g$, for all t, where ϕ_t is the time t flow of X. Show that X is Killing if and only if for all vector fields $X, Y \in \mathcal{X}(M)$

$$g(\nabla_Y X, Z) = -g(\nabla_Z X, Y).$$

Here ∇ is the Levi-Civita connection of g. Hint: Use that X is Killing if and only if $L_X g = 0$.

Problem #3. Given an n-dimensional Riemannian manifold (M, g). Show that at any point $p \in M$, there is a neighborhood U of p and vector fields $E_1, \ldots, E_n \in \mathcal{X}(U)$ so that

- $g(E_i, E_j) = \delta_{ij}$ – i.e., at each point of U, $E_1, \ldots, E_n$ form an orthonormal basis
- For $i, j = 1, \ldots, n$, $(\nabla_{E_i} E_j)_p = 0$ – i.e, at p the covariant derivative of each E_j is zero.

Problem #4. Fix a Riemannian manifold and a piece-wise C^1 curve $c : [a, b] \to M$. Let $P_t^c(v) \in T_{c(t)}M$ be the parallel transport of $v \in T_{c(a)}M$ to $T_{c(t)}M$ along c.

a) Show that the map $P_t^c : T_{c(a)}M \to T_{c(t)}M$ is a linear isometry between the inner product spaces $(T_{c(a)}M, g_{c(a)})$ and $(T_{c(t)}M, g_{c(t)})$.

b) For a fixed $p \in M$, let

$$\mathrm{Hol}_p = \{L : T_pM \to T_pM : L = P^c_{c(b)}, c : [a, b] \to M, c(a) = c(b) = p\}.$$

Here c is an admissible curve in M. Show that Hol_p is a subgroup of $O(T_pM, g_p)$, the set of linear isometries of (T_pM, g_p). For this reason it is called the holonomy group. (Hint: consider the natural group structure on the space of curves starting and ending at p).

c) What are the holonomy groups at each point of $(\mathbb{R}^n, \bar{g})$?

Problem #5. Determine the holonomy group at each point of $(\mathbb{S}^2, \mathring{g})$. (Hint: Use admissible curves that are triangles made up of pieces of great circles).