Math 645, Fall 2020: Assignment #6

Due: **Tuesday, October 27th**

Problem #1. Fix a manifold M. Two metrics g and h on M are *conformal* provided there is a function $u \in C^\infty(M)$ so that $h = e^{2u} g$.

a) If ∇^h is the Levi-Civita connection of h and ∇^g is the Levi-Civita connection of g show that for any $X, Y \in \mathcal{X}(M)$
$$\nabla^h_X Y = \nabla^g_X Y + (X \cdot u)Y + (Y \cdot u)X - g(X,Y)\mathrm{grad}^g u.$$
Here $\mathrm{grad}^g u$ is the g-gradient of u. (Hint: use compatibility with the metric).

b) Use this formula to determine the geodesics of the upper half-plane model of hyperbolic space. Recall, this is the Riemannian manifold $(\mathbb{U}^n, \breve{g})$ where $\mathbb{U}^n = \{x^n > 0\} \subset \mathbb{R}^n$ and with metric $\breve{g} = (x^n)^{-2}\bar{g}$.

Problem #2. Let (M, g) be a Riemannian manifold and $p \in M$. Suppose that $(x^1, \ldots, x^n)$ are geodesic normal coordinates at p (so $x^i(p) = 0$). Show that in these coordinates metric g satisfies the estimate

$$|g_{ij}(q) - \delta_{ij}| \leq C_1 r^2(q)$$

where q is near p and $C_1 > 0$ depends on g. Here

$$r(p) = \sqrt{(x^1(p))^2 + \ldots + (x^n(p))^2}.$$

is the euclidean distance to the origin in the coordinate chart. That is, in geodesic normal coordinates the euclidean metric approximates the metric g to first order.

Problem #3. Let (M, g) be a Riemannian manifold and $p \in M$. Use the preceding result to show that for $r > 0$ sufficiently small
$$|Vol_g(\mathcal{B}_r(p)) - \omega_n r^n| \leq C_2 r^{n+2}.$$
Here $\mathcal{B}_r(p)$ is the geodesic ball of radius r, ω_n is the volume of the unit ball in $\mathbb{R}^n$ and $C_2 > 0$ depends on g. You should use the fact that $\det(I_n + sA) = 1 + s\,\mathrm{tr}A + O(s^2)$.

Problem #4. Let (M, g) be a Riemannian manifold with Riemannian distance d_g.

a) Show that if $\gamma : (-\epsilon, \epsilon) \to M$ is a smooth curve, then
$$|\gamma'(0)|_g = \lim_{t \to 0} \frac{d_g(\gamma(t), 0)}{t}.$$

b) Show that if h is another Riemannian metric on M with Riemannian distance d_h and $d_g(p,q) = d_h(p,q)$ for all $p, q \in M$, then $g = h$. That is the Riemannian distance function uniquely determines the Riemannian metric.

Problem #5. Let (M, g) be a complete non-compact Riemannian manifold. Prove that for each $p \in M$, there is a ray $\gamma : [0, \infty) \to M$ starting from p. That is, $\gamma(0) = p$ and γ is a minimizing geodesic.