Math 645, Fall 2020: Assignment #7

Due: **Tuesday, November 3rd**

Problem #1. Let (M,g) be a Riemannian manifold and X a Killing vector field of (M,g) (recall a vector field is Killing if its flow is an isometry of g).

a) Show that the restriction of X to any geodesic gives a Jacobi field.

b) Show that if M is complete, then X is uniquely determined by knowing X_p and $(D_u X)_p$ for some $p \in M$ and all $u \in T_pM$.

Problem #2. Let (M,g) be a Riemannian manifold. Show that for $X, Y, Z \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$ one has $R(X,Y)(fZ) = fR(X,Y)Z$. This completes the proof that the curvature endomorphism is a tensor field.

Problem #3. Verify that the expression

$$sec(x,y) = \frac{R_p(x,y,y,x)}{|x|_g^2|y|_g^2 - g_p(x,y)^2}$$

depends only on the two-plane in T_pM spanned by $x, y \in T_pM$.

Problem #4. Show that if (M,g) has vanishing Riemann curvature tensor, then the map $\exp_p : U \to M$ is a local isometry between (U, g_p) and (M,g) where here U is a small neighborhood of 0 (this hypotheses is not needed when M is complete but simplifies the proof). Hint: Use the fact that the unit speed geodesics through p correspond to the lines with $x^i(\gamma_v(t)) = v^i t$ where $v = (v^1, \dots, v^n)$ satisfies $|v| = 1$. Next, consider the vector fields

$$\partial_r = \frac{1}{\sqrt{(x^1)^2 + \dots (x^n)^2}} \sum_{i=1}^n x^i \frac{\partial}{\partial x^i} \text{ and } W(i) = -x^i \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^i}$$

Check that $[\partial_r, W(i)] = 0$ and use this with $D_{\partial r}\partial_r = 0$ and the vanishing of the curvature to show $D_{\partial_r} W(i) = 1/rW(i)$ along γ_{e_1}. Use this to show that in fact $\frac{\partial}{\partial x^i}$ is parallel along γ_{e_1}.

Problem #5. Show that if (M,g) is a three dimensional Riemannian manifold, then the Ricci tensor uniquely determines the Riemann curvature tensor. Hint: This is a purely algebraic fact.