Math 645, Fall 2020: Assignment #8

Due: **Tuesday, November 10th**

Problem #1. Show that a three-dimensional Riemannian manifold $(M, g$ is Einstein if and only if it has constant sectional curvature.

Problem #2. Show that if $M \subset \mathbb{R}^{n+1}$ is a compact hypersurface (i.e., a codimension one submanifold), then there is a point $p \in M$ so that all the principal curvatures of M at p (with respect to some choice of unit normal) are positive. Hint: Consider the smallest euclidean ball centered at the origin containing M.

Problem #3. Show that there can be no C^2 ismoetric embedding

$$f : (\mathbb{T}^2, g^T) \to (\mathbb{R}^3, \bar{g}).$$

Here $g^T = \mathring{g} \oplus \mathring{g}$ is the (flat) product metric on $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. Hint: Use the previous exercise.

Problem #4. Let (M, g) be a Riemannian manifold. Let $\Omega \subset M$ be an open domain which is strongly convex (i.e. for every two points $p, q \in \Omega$ there is a minimizing geodesic contained in Ω connecting p to q) and so that $\partial\Omega$ is a smooth submanifold. Show that the principal curvatures of $\partial\Omega$ with respect to the inward pointing normal are non-negative.

Problem #5. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds and set $M = M_1 \times M_2$ with metric $g = g_1 \oplus g_2$, i.e., with the product metric.

a) Show that $M_1 \times \{p_2\} \subset M$ and $\{p_1\} \times M_2$ are totally geodesic submanifolds of M.
b) Show that $sec(\Pi) = 0$ for every two plane $\Pi \subset T_{(p_1,p_2)}M$ spanned by by $(v_1, 0)$ and $(0, v_2)$ for $v_1 \in T_{p_1}M$ and $v_2 \in T_{p_2}$.
c) Show that if (M_1, g) and (M_2, g) both have non-positive sectional curvature, then so does (M, g).