Math 645, Fall 2020: Assignment #9

Due: **Tuesday, December 1st**

Problem #1. Let (M, g) be a Riemannian manifold and $p \in M$. Show that the second order Taylor expansion of g in normal coordinates centered at p is

$$g_{ij}(x^1, \ldots, x^n) = \delta_{ij} - \frac{1}{3} \sum_{k,l=1}^{n} R_{iklj} x^l x^j + O(|x|^3)$$

Hint: Let $\gamma(t) = (tv^1, \ldots, tv^n)$ be a radial geodesic starting at p an $J(t) = tw^i \partial_i|_{\gamma(t)}$ be a Jacobi field along γ. Compute the first four derivatives of $|J(t)|^2$ at $t = 0$ in two different ways.

Problem #2. Let $\phi \in C^\infty([0, L))$ satisfy $\phi > 0$, $\phi^{(2k)}(0) = 0$ for all $k \geq 0$ (i.e., the derivatives of ϕ at 0 behave like those of an odd function) and $\phi'(0) = 1$ and consider the warped product metric $(M', g') = ((0, L)_r \times \mathbb{S}^n, \bar{g} \oplus \phi^2(r)\mathring{g})$.

a) Show that there is a $n + 1$ dimensional Riemannian manifold (M, g) and an isometric embedding $f : M' \to M$ so that $\lim_{r_i \to 0} f(r_i, v_i) = p_0 \in M$ exists and $f(M') = M \backslash \{p_0\}$.
b) Determine what the geodesics emanating from p_0 correspond to in M'.
c) Compute the sectional curvatures of (M, g) in terms of ϕ. Hint: Consider the Jacobi equation along geodesics emanating from p_0 – treat p_0 separately).

Problem #3.

a) Show that when n is odd any element of $A \in SO(n)$ has an eigenvector with eigenvalue 1.
b) Show that if (M, g) is oriented, then the parallel transport preserves the natural orientation induced on each T_pM.
c) Use the above to show that (M, g) is an even dimensional oriented Riemannian manifold and $c : [0, 1] \to M$ is a closed geodesic (i.e. c is geodesic, $c(0) = c(1)$ and $c'(0) = c'(1)$), then there is a $u \in T_{c(0)}$ orthogonal to $c'(0)$ so that the parallel transport, P along c satisfies $P(u) = u$.
d) Use the second variation formula to conclude that for every even dimensional oriented Riemannian manifold with positive sectional curvature, any closed geodesic must be unstable in that some variation decreases length (Hint: consider the variation obtained by parallel transporting the vector u.)
e) (Extra Credit): Use the fact that every free-homotopy class of a compact non-simply connected Riemannian manifold contains a geodesic that minimizes length in the class, to prove Synge's theorem: If (M, g) is a closed orientable Riemannian manifold with positive sectional curvature, then M is simply connected.

Problem #4. Prove *Wirtinger's inequality*: If $f : [0, \pi] \to \mathbb{R}$ is C^2 and satisfies $f(0) = f(\pi) = 0$, then

$$\int_0^\pi f^2 dt \leq \int_0^\pi (f')^2 dt,$$

with equality if and only if $f(t) = c \sin(t)$ where $c \in \mathbb{R}$. Hint: Use Fourier series or the calculus of variations.

Problem #5. Let M be a complete simply connected Riemannian manifold. For each $p \in M$, denote by $\text{conj}(p)$, the set of first conjugate points of p (i.e., for each $v \in T_pM$ with $v \neq 0$, consider the first conjugate point of the geodesic $c_v(s) = \exp_p(sv)$).

a) Show that if $p' \in \text{conj}(p)$ satisfies $d_g(p, p') = L$, then there is a unit speed geodesic $\gamma : [0, L] \to M$ connecting p to p', so that there is a non-trivial Jacobi field J along γ so $J(0) = J(L) = 0$, and $I(J, J) = 0$.
b) If for each $p \in M$, $\text{conj}(p)$ consists of one point p' with $d_g(p, p') = d_g(p, \text{conj}(p)) = \pi$ and the sectional curvature of M satisfies $0 < \delta \leq K \leq 1$, then M has constant sectional curvature 1. Hint: Consider the geodesic, γ, and Jacobi field, J, from part a), using the Jacobi equation, the index property of J and the previous problem conclude that along the geodesic γ one has $K(\gamma', J) = 1$, where $K(\gamma', J)$ is the sectional curvature of the two plane spanned by γ' and J.