Math 645, Fall 2017: Assignment #3

Due: **Thursday, October 5th**

Problem #1. Let (M,g) and (N,h) be Riemannian manifolds and f a smooth positive function on M. Define a $(0,2)$-tensor on $M \times N$ by

$$(g \times_f h)_p(X_p, Y_p) = g_{\pi_M(p)}(T_p\pi_M(X_p), T_p\pi_M(Y_p)) + f^2(\pi_M(p))h_{\pi_N(p)}(T_p\pi_N(X_p), T_p\pi_N(Y_p))$$

where here π_N, π_M are the natural projections.

a) Show that $g \times_f h$ is a Riemannian metric on $M \times N$. It is called a *warped product metric*.
b) Let $(M,g) = (\mathbb{R}^+, g^E)$ and $(N,h) = (\mathbb{S}^n, g^S)$ and denote by $r \in C^\infty(M)$ the standard coordinate on $\mathbb{R}^+$. Consider the following family of warped product metrics

$$c_\lambda = g \times_{\lambda r} h.$$

Show that $(\mathbb{R}^+ \times \mathbb{S}^1, c_\lambda)$ is locally isometric to $(\mathbb{R}^2 \backslash \{0\}, g^E)$ for all $\lambda > 0$.
c) Show that $(\mathbb{R}^+ \times \mathbb{S}^1, c_\lambda)$ is isometric to $(\mathbb{R}^2 \backslash \{0\}, g^E)$ if and only if $\lambda = 1$. (Hint: Consider parallel transportation around $\pi_{\mathbb{R}^+}^{-1}(1)$).

Problem #2.

a) Let $(M,g) = ((0,2\pi), g^E)$ and $(N,h) = (\mathbb{S}^n, g^S)$ show that

$$g^E \times_{\sin(r)} g^S$$

is locally isometric to $(\mathbb{S}^{n+1}, g^S)$.
b) Let $(M,g) = (\mathbb{R}^+, g^E)$ and $(N,h) = (\mathbb{S}^n, g^S)$ show that

$$g^E \times_{\sinh(r)} g^S$$

is locally isometric to $(\mathbb{H}^{n+1}, g^H)$.

Problem #3. Verify that a $\phi \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ that satisfies $\phi(0) = 0$, is an isometry of $(\mathbb{R}^n, g^E)$ if and only if ϕ can be identified with an element of $O(n) = \{A \in \mathbb{R}^{n\times n} : A^\top A = I_n\}$.

Problem #4. Let (M,g) be a closed Riemannian manifold. We say a vector field $X \in \mathcal{X}(M)$ is a *Killing field*, provided $\phi_t^* g = g$, for all t, where ϕ_t is the time t flow of X. Show that X is Killing if and only if for all vector fields $X, Y \in \mathcal{X}(M)$

$$g(\nabla_Y X, Z) = -g(\nabla_Z X, Y).$$

Here ∇ is the Levi-Civita connection of g. Hint: Use that X is Killing if and only if $L_X g = 0$.

Problem #5. Define a connection ∇ on $\mathbb{R}^3$ by

$$\begin{aligned}
&\nabla_{\partial_{x_1}}\partial_{x_1} = \nabla_{\partial_{x_2}}\partial_{x_2} = \nabla_{\partial_{x_3}}\partial_{x_3} = 0, && \\
&\nabla_{\partial_{x_1}}\partial_{x_2} = \partial_{x_3}, && \nabla_{\partial_{x_2}}\partial_{x_1} = -\partial_{x_3}, \\
&\nabla_{\partial_{x_1}}\partial_{x_3} = -\partial_{x_2}, && \nabla_{\partial_{x_3}}\partial_{x_1} = \partial_{x_2}, \\
&\nabla_{\partial_{x_2}}\partial_{x_3} = \partial_{x_1}, && \nabla_{\partial_{x_3}}\partial_{x_2} = -\partial_{x_1},
\end{aligned}$$

where here $\partial_{x_i} = \frac{\partial}{\partial x^i}$ are the coordinate vector fields of the standard coordinates.

a) Verify that this connection is compatible with g^E but is not torsion-free.
b) Determine the zero-acceleration curves of ∇.
c) Determine the parallel transport of ∇ along zero-acceleration curves.