Math 645, Fall 2017: Assignment #4

Due: **Thursday, October 12th**

Problem #1. Fix a Riemannian manifold and a piece-wise C^1 curve $c : [a, b] \to M$. Let $P_t^c(v) \in T_{c(t)}M$ be the parallel transport of $v \in T_{c(a)}M$ to $T_{c(t)}M$ along c.

a) Show that the map $P_t^c : T_{c(a)}M \to T_{c(t)}M$ is a linear isometry between the inner product spaces $(T_{c(a)}M, g_{c(a)})$ and $(T_{c(t)}M, g_{c(t)})$.
b) For a fixed $p \in M$, let

$$\mathrm{Hol}_p = \{L : T_pM \to T_pM : L = P^c_{c(b)}, c : [a, b] \to M, c(a) = c(b) = p\}.$$

Show that Hol_p is a subgroup of $O(T_pM, g_p)$, the set of linear isometries of (T_pM, g_p). For this reason it is called the holonomy group. (Hint: consider the natural group structure on the space of curves starting and ending at p).
c) What are the holonomy groups at each point of $(\mathbb{R}^n, g^E)$?

Problem #2.

a) Determine the holonomy group at each point of $(\mathbb{S}^2, g^S)$.
b) Determine the holonomy group at each point of $(\mathbb{RP}^2, g)$ where g is the metric induced by g^S.

Problem #3. Let $M = \mathbb{R}^2 \backslash \bar{B}_1$ where here $\bar{B}_1 = \{(x^1)^2 + (x^2)^2 \leq 1\}$ and so M is an open subset of $\mathbb{R}^2$. Consider the Riemannian manifold (M, g), where g is the metric inherited from g^E.

a) Describe the exponential map at $(2, 0)$.
b) Determine the distance, in (M, g), between $(2, 0)$ and $(-2, 0)$.

Problem #4. Let $M \subset \mathbb{R}^n$ be a submanifold and let g be the Riemannian metric on M induced by g^E. Show that for any points $p, q \in M$ that $d_{g^E}(p, q) \leq d_g(p, g)$. Suppose $p, q \in M$ are connected, in M, by a length minimizing geodesic, γ (i.e., γ a geodesic of g), what must be true of γ if $d_g(p,) = L_g(\gamma) = d_{g^E}(p, q)$.

Problem #5. Given an n-dimensional Riemannian manifold (M, g). Show that at any point $p \in M$, there is a neighborhood U of p and vector fields $E_1, \ldots, E_n \in \mathcal{X}(U)$ so that

- $g(E_i, E_j) = \delta_{ij}$ – i.e., at each point of U, $E_1, \ldots, E_n$ form an orthonormal basis
- For $i, j = 1, \ldots, n$, $(D_{E_i} E_j)_p = 0$ – i.e, at p the covariant derivative of each E_j is zero.