Math 645, Fall 2017: Assignment #5

Due: **Thursday, October 19th**

Problem #1. Let (M, g) be a Riemannian manifold and $p \in M$. Suppose that $(x^1, \ldots, x^n)$ are geodesic normal coordinates at p (so $x^i(p) = 0$). Show that in these coordinates metric g satisfies the estimate

$$|g_{ij}(q) - \delta_{ij}| \leq C_1 r^2(q)$$

where q is near p and $C_1 > 0$ depends on g. Here

$$r(p) = \sqrt{(x^1(p))^2 + \ldots + (x^n(p))^2}.$$

is the euclidean distance to the origin in the coordinate chart. That is, in geodesic normal coordinates the euclidean metric approximates the metric g to first order.

Problem #2. Let (M, g) be a Riemannian manifold and $p \in M$. Use the preceding result to show that for $r > 0$ sufficiently small

$$|Vol_g(\mathcal{B}_r(p)) - \omega_n r^n| \leq C_2 r^{n+2}.$$

Here $\mathcal{B}_r(p)$ is the geodesic ball of radius r, ω_n is the volume of the unit ball in $\mathbb{R}^n$ and $C_2 > 0$ depends on g. You should use the fact that $\det(I_n + sA) = 1 + s \operatorname{tr} A + O(s^2)$.

Problem #3. Consider $(\mathbb{S}^n, g^S)$. Arguing geometrically, show that if $n \geq 2$, then for all $r > 0$

$$Vol_{g^S}(\mathcal{B}_r(p)) < \omega_n r^n.$$

What happens when $n = 1$?

Problem #4. Let (M, g) and (N, h) be Riemannian manifolds. Show that if $\phi : M \to N$ is a Riemannian isometry (i.e., it is a diffeomorphism and $\phi^* h = g$), then it is a metric isometry (i.e., $d_h(\phi(p), \phi(q)) = d_g(p, q)$.

Problem #5. Let (M, g) be a complete non-compact Riemannian manifold. Prove that for each $p \in M$, there is a ray $\gamma : [0, \infty) \to M$ starting from p. That is, $\gamma(0) = p$ and γ is a minimal geodesic.