Math 645, Fall 2017: Assignment #8

Due: **Tuesday, November 14th**

Problem #1. Let $\phi \in C^\infty([0, L))$ satisfy $\phi > 0$, $\phi^{(2k)}(0) = 0$ for all $k \geq 0$ (i.e., the derivatives of ϕ at 0 behave like those of an odd function) and $\phi'(0) = 1$ and consider the warped product metric $(M', g') = ((0, L) \times \mathbb{S}^n, g^E \times_\phi g^S)$.

a) Show that there is a $n+1$ dimensional Riemannian manifold (M, g) and an isometric embedding $f : M' \to M$ so that $\lim_{r_i \to 0} f(r_i, v_i) = p_0 \in M$ exists and $f(M') = M \backslash \{p_0\}$.
b) Determine what the geodesics emanating from p_0 correspond to in M'.
c) Compute the sectional curvatures of (M, g) in terms of ϕ. (Hint: The Jacobi equation along geodesics emanating from p_0 – treat p_0 separately).)

Problem #2. Let (M, g) be a complete Riemannian manifold of dimension n with non-positive curvature. Let X be a Killing vector field on M.

a) Show that if X has two distinct zeros, then X must vanish on any geodesic joining the two zeros. (Hint: Use last weeks homework.)
b) (Bonus): Show that when $n = 2$, if X admits two distinct zeros, then it must vanish identically. (Hint: Use the identity from Homework 3 and the result from last week)
c) (Bonus): Show by example that X may have two distinct zeros but not vanish identically when $n \geq 3$.

Problem #3. Fix a manifold M. Two metrics g and h on M are *conformal* provided there is a function $u \in C^\infty(M)$ so that $h = e^{2u} g$.

a) If D^h is the Levi-Civita connection of h and D^g is the Levi-Civita connection of g show that for any $X, Y \in \mathcal{X}(M)$
$$D^h_X Y = D^g_X Y + (X \cdot u)Y + (Y \cdot u)X - g(X, Y)\nabla_g u.$$
Here $\nabla_g u$ is the g-gradient of u. (Hint: use compatibility with the metric).
b) Use this formula to determine the geodesics of the upper half-plane model of hyperbolic space. Recall, this is the Riemannian manifold $(\mathbb{H}^n, g^H)$ where $\mathbb{H}^n = \{x^n > 0\} \subset \mathbb{R}^n$ and with metric $g^H = (x^n)^{-2} g^E$.

Problem #4. Given two Riemannian manifolds (M, g) and (N, h) we say a map $f : M \to N$ is *conformal* if $f^* h$ is conformal to g.

a) Show that the map $f : U \to \mathbb{R}^n$ where $U \subset \mathbb{R}^n$ given by
$$f(x) = p_0 + \frac{\lambda A(x - p_1)}{|x - p_1|^\epsilon}$$
is conformal from (U, g^E) to $(\mathbb{R}^n, g^E)$ when $\lambda > 0$, $p_0 \in \mathbb{R}^n$, $p_1 \in \mathbb{R}^n \backslash U$, $\epsilon = 0, 2$ and $A \in O(n)$. That is compositions of translation, rotation, homothetic scaling and "inversion" are conformal.
b) Use complex analysis to give a conformal map not of this form when $n = 2$.
c) (Bonus): Show that when $n \geq 3$ the only conformal maps are those found in a). This is called Liouville's theorem.

Problem #5. Show that if $(\mathbb{R}^2, g)$ is a complete Riemannian manifold, then
$$\lim_{r \to \infty} \inf_{(x^1)^2 + (x^2)^2 \geq r^2} S(x^1, x^2) \leq 0.$$
Here (x^1, x^2) are the standard coordinates on $\mathbb{R}^2$ and $S(x^1, x^2)$ is the scalar curvature at the point (x^1, x^2).