PDE Undergrad

HW1 Solution

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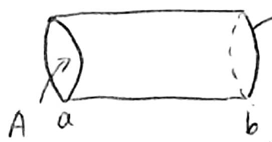
[1.2.9] Consider a thin 1D rod without sources of thermal energy whose lateral surface area is not insulated.

(a) Assume that the heat energy flowing out of the lateral sides per unit surface area per unit time is $w(x,t)$. Derive the PDE for the temperature $u(x,t)$

~~tot~~ P = lateral perimetre



Sol:
$$\frac{d}{dt}\int_a^b eA\,dx = A(\phi(a,t)-\phi(b,t)) - P\int_a^b w\,dx + \underbrace{A\int_a^b Q\,dx}_{} = 0. \text{ (no sources)}$$

$$\Rightarrow A\int_a^b \partial_t e\,dx = -A\int_a^b \partial_x\phi\,dx - P\int_a^b w\,dx$$

$$\overset{e=c\rho u}{\Rightarrow} c\rho\int_a^b \partial_t u\,dx = -\int_a^b \left(\partial_x\phi + \frac{P}{A}w\right)dx.$$

$$\Rightarrow \int_a^b \left(c\rho\,\partial_t u + \partial_x\phi + \frac{P}{A}w\right)dx = 0 \Rightarrow c\rho\,\partial_t u = -\partial_x\phi - \frac{P}{A}w$$

Now $\phi = -k_0\,\partial_x u$ so we have $c\rho\,\partial_t u - k_0\,\partial_x^2 u = -\frac{P}{A}w(x,t)$

(b). Assume $w(x,t)$ is proportional to the temperature difference between the rod $u(x,t)$ and a known outside temperature $\gamma(x,t)$. Derive that

$$c\rho\,\partial_t u = k_0\,\partial_x^2 u - \frac{P}{A}(u(x,t)-\gamma(x,t))\,h(x).$$

Sol: ~~Replace~~ Insert $w(x,t) = (u(x,t)-\gamma(x,t))\,h(x)$ into (a)

$\square$

[1.4.1] Determine the equilibrum temperature distribution for a 1D rod with const thermal properties with the followiy sources and bdry conditions

(g) $\begin{cases} \partial_t u = k\,\partial_x^2 u + Q \\ Q = 0 \\ u(0) = T \quad \cancel{\partial_x u(L) = 0} \\ \partial_x u(L) + u(L) = 0 \end{cases}$

Sol: Equilibrum $\Rightarrow \partial_t u = 0$.

So $k_0\, \partial_x^2 u = 0 \Rightarrow u(x) = Ax + B$

Pluggiy in the bdry conditions we have

$\begin{cases} B = T \\ A + AL + B = 0 \end{cases} \Rightarrow \begin{cases} A = -\frac{T}{1+L} \\ B = T \end{cases}$

$\Rightarrow u(x) = -\frac{T}{1+L} x + T$ □

[1.4.11] Suppose $\begin{cases} \partial_t u = \partial_x^2 u + x \\ u(x,0) = f(x) \\ \partial_x u(0,t) = \beta, \quad \partial_x u(L,t) = 7 \end{cases}$

(1) Calculate the total thermal energy in the 1D rod.

Sol: $E(t) = \int_0^L e A\, dx = c\rho A \int_0^L u(x,t)\, dx$

$c\rho = k_0 = 1$

↑ from $\partial_t u = \frac{k_0}{c\rho} \partial_x^2 u + \frac{Q(x,t)}{c\rho}$

$$\frac{d}{dt}\int_0^L u(x,t)\,dx = \int_0^L \partial_t u(x,t)\,dx$$
$$= \int_0^L \partial_x^2 u(x,t)\,dx + \int_0^L x\,dx$$
$$= \partial_x u \Big|_{x=0}^{x=L} + \frac{1}{2}L^2$$
$$= \partial_x u(L,t) - \partial_x u(0,t) + \frac{1}{2}L^2$$
$$= 7 - \beta + \frac{1}{2}L^2$$

$$\therefore \int_0^L u(x,t)\,dx = \int_0^L u(x,0)\,dx + \int_0^t \frac{d}{ds}\left(\int_0^L u(x,s)\,dx\right) ds$$
$$= \int_0^L f(x)\,dx + \left(7 - \beta + \frac{1}{2}L^2\right)t$$

$$\Rightarrow E(t) = \left(\int_0^L f(x)\,dx + \left(7 - \beta + \frac{1}{2}L^2\right)t\right)$$

(2) Determine a value β for which an equilibrum exists and $\lim_{t\to\infty} u(x,t) = ?$

equilibrum $\Rightarrow \partial_t E = 0 \Rightarrow 7 - \beta + \frac{1}{2}L^2 = 0 \quad \therefore \beta = 7 + \frac{1}{2}L^2$

For the equilibrum $\partial_t u = 0$

$\Rightarrow \partial_x^2 u + x = 0$

$\Rightarrow u''(x) = -x \Rightarrow u(x) = -\frac{1}{6}x^3 + C_1 x + C_2$

integrate w.r.t x twice

Insert this into the bdry condition $\begin{cases} \partial_x u(0) = \beta \\ \partial_x u(L) = 7 \end{cases}$

$\Rightarrow C_1 = \frac{1}{2}L^2 + 7 = \beta$

$\therefore E(t) = \int_0^L c\rho\, u(x)\, dx$

$= \int_0^L \left(-\frac{1}{6}x^3 + (7 + \frac{1}{2}L^2)x + C_2\right) dx$

$= \frac{5}{24}L^4 + \frac{7}{2}L^2 + C_2 L = \int_0^L f(x)\,dx$

So $C_2 = \frac{1}{L}\int_0^L f(x)\,dx - \frac{5}{24}L^3 - \frac{7}{2}L$

$\therefore \lim_{t\to\infty} u(x,t) = -\frac{1}{6}x^3 + (7 + \frac{1}{2}L^2)x + \frac{1}{L}\int_0^L f(x)\,dx - \frac{5}{24}L^3 - \frac{7}{2}L$ $\square$

[1.5.3] Consider polar coordinates $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

(1) Find $\partial_x r$, $\partial_y r$, $\partial_x\theta$, $\partial_y\theta$.

Sol: $r^2 = x^2 + y^2 \Rightarrow 2r\,\partial_x r = 2x \Rightarrow \partial_x r = \frac{x}{r} = \cos\theta$.

Similarly one has $\partial_y r = \sin\theta$.

$x = r\cos\theta \Rightarrow 1 = \partial_x r \cos\theta + r\,\partial_x(\cos\theta)$

$= \partial_x r \cos\theta - r\sin\theta\,\partial_x\theta = \cos^2\theta - r\sin\theta\,\partial_x\theta$

$\Rightarrow r\sin\theta\,\partial_x\theta = \cos^2\theta - 1 = -\sin^2\theta$

$\Rightarrow \partial_x\theta = -\frac{\sin\theta}{r}$

Similarly one has $\partial_y\theta = \frac{\cos\theta}{r}$.

(2) Show that $\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$r\hat{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} \Rightarrow \hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

Since $\begin{cases} \hat{r} \perp \hat{\theta} \\ \|\hat{\theta}\|^2 = 1 \end{cases} \Rightarrow \hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

i.e. $\hat{\theta}$ is the unit vector perpendicular to $\hat{r}$.

(3) Show that $\nabla = \hat{r}\,\partial_r + \hat{\theta}\,\frac{1}{r}\,\partial_\theta$.

Sol: $\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\cdot\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}$ (chain rule)

$$= \cos\theta\,\partial_r - \frac{\sin\theta}{r}\,\partial_\theta$$

Similarly $\frac{\partial}{\partial y} = \sin\theta\,\partial_r + \frac{\cos\theta}{r}\,\partial_\theta$

$$\nabla = \partial_x\,\hat{i} + \partial_y\,\hat{j} = (\cos\theta\,\hat{i} + \sin\theta\,\hat{j})\,\partial_r + \frac{1}{r}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\,\partial_\theta$$

$$= \hat{r}\,\partial_r + \frac{1}{r}\,\hat{\theta}\,\partial_\theta$$

(4) If $A = A_r\hat{r} + A_\theta\hat{\theta}$, show that $\nabla\cdot A = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(A_\theta)$

Sol: $\nabla\cdot A = \left(\hat{r}\,\partial_r + \frac{1}{r}\hat{\theta}\,\partial_\theta\right)\cdot\left(A_r\,\hat{r} + A_\theta\,\hat{\theta}\right)$

$$\overset{\hat{r}\cdot\hat{\theta}=0}{=} \hat{r}\,\partial_r(A_r\hat{r}) + \frac{1}{r}\hat{\theta}\cdot\partial_\theta(A_r\hat{r}) + \hat{r}\,\partial_r(A_\theta\hat{\theta}) + \frac{\hat{\theta}}{r}\,\partial_\theta(A_\theta\hat{\theta})$$

$$\overset{\text{use (2)}}{=} \partial_r A_r + \frac{1}{r}A_r + \frac{1}{r}\partial_\theta A_\theta$$

$$= \frac{1}{r}\,\partial_r(rA_r) + \frac{1}{r}\,\partial_\theta A_\theta$$

(5) Show that $\Delta u := \nabla^2_d u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$

$$\Delta u = \cancel{\mathrm{div}(\nabla u)} = \nabla\cdot(\nabla u) \longrightarrow$$

$$\boxed{\begin{aligned}\Delta u &= \partial_1^2 u + \dots + \partial_d^2 u \quad \text{in } \mathbb{R}^d.\\ &= \underbrace{(\partial_1,\dots,\partial_d)}_{\nabla}\cdot\underbrace{(\partial_1 u,\dots,\partial_d u)}_{\nabla u}\\ &= \nabla\cdot(\nabla u)\\ &\text{i.e.} = \mathrm{div}(\mathrm{grad}\,u).\end{aligned}}$$

let $A = \nabla u$ in (4) $\Rightarrow$

$$= \frac{1}{r}\,\partial_r\Big(r\cdot\underbrace{(\cancel{\hat{r}}\,\partial_r u)}_{=(\nabla u)_r}\cancel{+\frac{1}{r}\hat{\theta}\,\partial_\theta u}\Big)$$

$$+ \frac{1}{r}\,\partial_\theta\Big(\underbrace{\frac{1}{r}\,\cancel{\hat{\theta}}\,\partial_\theta u}_{(\nabla u)_\theta}\Big)$$

$$= \frac{1}{r}\,\partial_r(r\,\partial_r u) + \frac{1}{r^2}\,\partial_\theta^2 u$$

□