PDE Undergrad

HW4 Solution

Junyan Zhang
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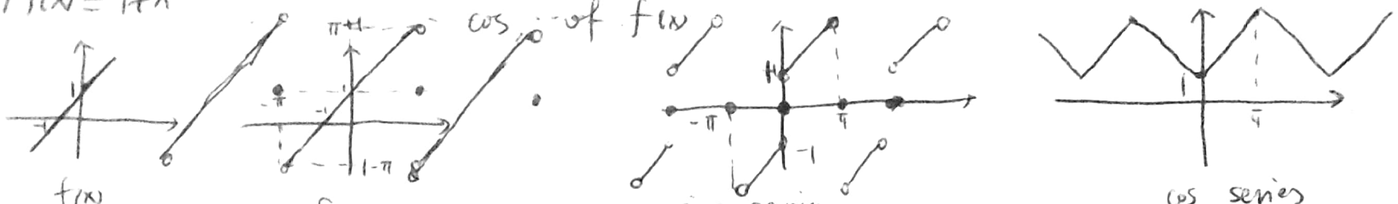
[3.3] 1. Sketch f(x), Fourier series of f(x), Fourier sin … of f(x), cos … of f(x)

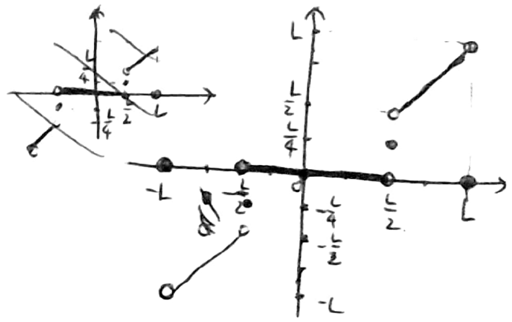
(b) $f(x) = 1 + x$



□

2. Sketch the Fourier ~~Series~~ Sine Series of f(x) and determine coefficients

(c) $f(x) = \begin{cases} 0 & x < \frac{L}{2} \\ x & x > \frac{L}{2} \end{cases}$



$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\,dx$

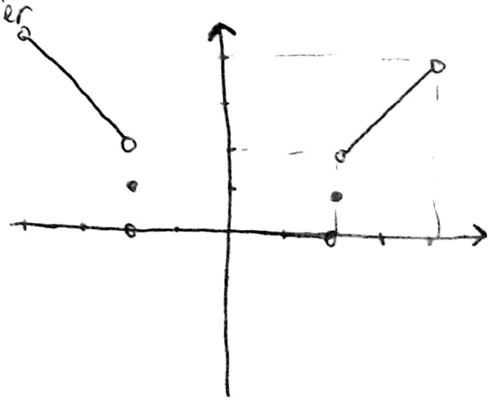
$= \frac{2}{L}\int_{\frac{L}{2}}^{L} x \sin\frac{n\pi x}{L}\,dx \overset{y=\frac{x}{L}}{\underset{dx=Ldy}{=}} 2L\int_{\frac{1}{2}}^{1} y \sin(n\pi y)\,dy$

$= \frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{2L}{n\pi}\underbrace{(-1)^n}_{\cos n\pi} - \frac{2L}{(n\pi)^2}\sin\frac{n\pi}{2}$

□

5. Sketch the Fourier cosine Series of

(c) $f(x) = \begin{cases} 0 & x < \frac{L}{2} \\ x & x > \frac{L}{2} \end{cases}$ and determine Fourier coefficients



$a_n = \frac{2}{L}\int_{2/L}^{L} x \cos\frac{n\pi x}{L}\,dx$

$= -\frac{L}{n\pi}\sin\frac{n\pi}{2} + \frac{2L}{(n\pi)^2}(-1)^n - \frac{2L}{(n\pi)^2}\cos\frac{n\pi}{2}$

□

$a_0 = 3L/8$

18. For continuous functions — This condition should be C^1 functions (continuous + continuous 1st order derivative)

(1) Under what conditions does $f(x)$ equal to its Fourier series $\forall x$ $-L \le x \le L$

$f(L) = f(-L)$. f cont.

(2) $f(x) \stackrel{?}{=}$ sine series?

$f(0) = f(L) = 0$. f cont.

(3) $f(x) \stackrel{?}{=}$ cos series?

No extra condition needed.

Rmk: For f only continuous there exists some counterexample s.t. the Fourier series of f nowhere converges to f !!!

[3.4] 1. (a). $\int_a^b u \frac{dv}{dx}\, dx = \int_a^{c-} + \int_{c+}^{b}$ $\square$

integrate by parts

$$= u(c-)v(c-) - u(a)v(a) + u(b)v(b) - u(c+)v(c+)$$
$$- \int_a^b v \frac{du}{dx}$$
$$= uv\Big|_a^b - uv\Big|_{c-}^{c+} - \int_a^b v \frac{du}{dx}\, dx$$

(b). when $u(c-) = u(c+)$, $v(c-) = v(c+)$ this automatically becomes I.B.P. $\square$

6. step 2 $\not\Rightarrow$ step 3 $\quad e^x = -\sum_n \frac{n\pi}{L} A_n \sin\frac{n\pi x}{L}$ not continuous for RHS b/c $e^0 = 1$.

$\downarrow$ one cannot differentiate once more.

Instead; The cosine Fourier series of e^x

$$e^x \sim \frac{1}{L}(f(L) - f(0)) + \sum_{n\ge 1}\Big(\underbrace{\frac{n\pi}{L} B_n}_{= -\frac{n\pi}{L} A_n} + \frac{2}{L}\big((-1)^n f(L) - f(0)\big)\Big)\cos\frac{n\pi x}{L}$$

Thus $A_0 = \frac{1}{L}(f(L) - f(0)) = \frac{1}{L}(e^L - 1)$

$$A_n = -\frac{n^2\pi^2}{L^2} A_n + \frac{2}{L}\left((-1)^n e^L - 1\right)$$

$$\Rightarrow A_n = \frac{2}{L} \cdot \frac{(-1)^n e^L - 1}{1 + \frac{n^2\pi^2}{L^2}}$$ $\square$

7. Prof: $u(x,t) = \frac{a_0(t)}{2} + \sum_{n=\phi}^{\infty} a_n(t)\cos\frac{n\pi x}{L} + b_n(t)\sin\frac{n\pi x}{L}$

$$a_0(t) = \frac{1}{2L}\int_{-L}^{L} u(x,t)\,dt$$

$$a_n(t) = \frac{1}{L}\int_{-L}^{L} u(x,t)\cos\frac{n\pi x}{L}\,dx$$

Assume $\partial_t u(x,t) \sim \hat{a}_0(t) + \sum_{n\geq 1}\left(\hat{a}_n(t)\cos\frac{n\pi x}{L} + \hat{b}_n(t)\sin\frac{n\pi x}{L}\right)$

then $\hat{a}_0(t) = \frac{1}{2L}\int_{-L}^{L}\partial_t u\,dx = \partial_t\left(\frac{1}{2L}\int_{-L}^{L} u(x,t)\,dx\right) = a_0'(t)$

$$\hat{a}_n(t) = \frac{1}{L}\int_{-L}^{L}\partial_t u\cdot\cos\frac{n\pi x}{L}\,dx = \partial_t\left(\frac{1}{L}\int_{-L}^{L} u\cdot\cos\frac{n\pi x}{L}\,dx\right) = a_n'(t)$$

similarly $\hat{b}_n(t) = b_n'(t)$

□