Undergrad PDE

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Hw6 Solution

[4.2] 1 (a) Using equ.(4.2.7), to compute the ~~sg~~ sagged equilibrium position $u_E(x)$ if $Q(x,t)=-g$. The bdry conditions are $u(0)=0$ and $u(L)=0$

Sol: If $Q(x,t)=-g$ then:
$$\begin{cases} T_0\,\partial_x^2 u = g\rho_0 \\ u(0)=u(L)=0 \end{cases}$$

Integrating in $x \Rightarrow \partial_x u_E(x) - \partial_x u_E(0) = \frac{g}{T_0}\int_0^x \rho_0(u)\,du + C_1$

Integrating in $x$ again $\Rightarrow u_E(x) - u_E(0) = \partial_x u_E(0) + \frac{g}{T_0}\int_0^x\int_0^{\xi}\rho_0(u)\,du\,d\xi + C_1x + C_2$

$u_E(0) = C_2 = 0$

$u_E(L) = \partial_x u_E(0)\cdot L + \frac{g}{T_0}\int_0^L\int_0^{\xi}\rho_0(u)\,du\,d\xi + C_1 L = 0$

$\Rightarrow C_1 = -\partial_x u_E(0) - \frac{g}{T_0 L}\int_0^L\int_0^{\xi}\rho_0(u)\,du\,d\xi$

$\Rightarrow u_E(x) = \partial_x u_E(0) + \frac{g}{T_0}\int_0^x\int_0^{\xi}\rho_0(u)\,du\,d\xi + C_1 x$ $\square$

[4.3] (1). (b) is more likely ~~to be~~ correct.

~~$u(L,t) - u_E(t) > 0 \Rightarrow \partial_x u(L,t) < 0$~~ 4.3.5

$\underset{>0}{T_0}\,\partial_x u(L,t) = -\underset{>0}{K}\,u(L,t)$ $u(L,t)>0 \Rightarrow \partial_x u(L,t)<0$ only (b) satisfies this condition.

[4.4]3. $\rho_0 \partial_t^2 u = T_0 \partial_x^2 u - \beta \partial_t u$

(1) Briefly explain why $\beta > 0$

(2) Determine the sol. that satisfies the bdry conditions $u(0,t)=0$, $u(L,t)=0$

w/ initial data $u(x,0)=f(x)$, $\partial_t u(x,0) = g(x)$

(Assume $\beta^2 < \frac{4\pi^2 \rho_0 T_0}{L^2}$

sol: (1). $\partial_t u$ = velocity of the string. The damping force should be always opposite to the direction of motion and proportional to the velocity. $\Rightarrow \beta > 0$

(2) let $u(x,t) = X(x)T(t)$

$$\Rightarrow \rho_0 X T'' = T_0 X'' T - \beta X T'$$

$$\Rightarrow \frac{X''}{X} = \frac{\rho_0 T'' + \beta T'}{T_0 T} = -\lambda$$

$\Rightarrow$ We get 2 ODEs:

① $\begin{cases} X''(x) = -\lambda X(x) \\ X(0) = X(L) = 0 \end{cases}$ $\Rightarrow X_n(x) = C_n \sin\frac{n\pi}{L}x$ with $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n \geq 1$

② $\frac{\rho_0 T'' + \beta T'}{T_0 T} = -\left(\frac{n\pi}{L}\right)^2$ $n \geq 1$

$\Rightarrow \rho_0 T'' + \beta T' + \left(\frac{n\pi}{L}\right)^2 T_0 T = 0$, whose determinant $\Delta_n = \frac{4n^2\pi^2}{L^2}\rho_0 - \beta^2 > 0$

$$\Rightarrow T_n = e^{-\frac{\beta}{2\rho_0}t}\left(A_n \cos\left(\frac{\sqrt{\Delta_n}}{2\rho_0}t\right) + B_n \sin\left(\frac{\sqrt{\Delta_n}}{2\rho_0}t\right)\right)$$

· Therefore, $u(x,t) = \sum_{n=1}^{\infty} e^{-\frac{\beta}{2\rho_0}t}\left(A_n \cos\left(\frac{\sqrt{\Delta_n}}{2\rho_0}t\right) + B_n \sin\left(\frac{\sqrt{\Delta_n}}{2\rho_0}t\right)\right) \sin\frac{n\pi x}{L}$

· From $u(x,0) = f(x)$, $\partial_t u(x,0) = g(x)$ $\Rightarrow$ $A_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$

$$B_n = \frac{4\rho_0}{\sqrt{\Delta_n} L}\int_0^L (g(x) + \beta f(x)) \sin\frac{n\pi x}{L}\, dx.$$

$u(x,0)=f(x) \Rightarrow f(x)=\sum_{n\geq 1}^{\infty} A_n \sin\frac{n\pi x}{L} \quad \Rightarrow A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

$$\overset{\parallel}{\sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\right)\sin\frac{n\pi x}{L}}$$

$$\partial_t u(x,0)=g(x) \Rightarrow \sum_{n=1}^{\infty}\left(-\frac{\beta}{2\rho_0}A_n + B_n\frac{\sqrt{\Delta_n}}{2\rho_0}\right)\sin\frac{n\pi x}{L}$$

$$\overset{\parallel}{\sum_{n=1}^{\infty}\underbrace{\left(\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx\right)}\sin\frac{n\pi x}{L}}$$

$$\Rightarrow B_n = \frac{2\rho_0}{\sqrt{\Delta_n}}\left(\frac{\beta}{2\rho_0}A_n + \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx\right)$$

$$= \frac{2\rho_0\cdot 2}{\sqrt{\Delta_n}\cdot L}\int_0^L\left(f(x)\frac{\beta}{2\rho_0}+g(x)\right)\sin\frac{n\pi x}{L}\,dx.$$

$$\frac{4\rho_0}{\sqrt{\Delta_n}L} =$$

□

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4. Redo 4.4.3(b) by eigenfunction expansion

Let $u(x,t) = \cancel{\sum_{n=1}^{\infty} B_n \frac{\sin n\pi x}{L} B_n \cos n\pi}$

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

$$\rho_0\partial_t^2 u = T_0\partial_x^2 u - \beta\partial_t u \Rightarrow \rho_0 B_n''(t) + T_0\left(\frac{n\pi}{L}\right)^2 B_n(t) - \beta B_n'(t) = 0$$

$$\Delta_n = \frac{4n^2\pi^2}{L^2}\rho_0 - \beta^2 > 0$$

$$\Rightarrow B_n(t) = e^{-\frac{\beta}{2\rho_0}t}\left(A_n\cos\frac{\sqrt{\Delta_n}}{2\rho_0}t + \tilde{A}_n\sin\frac{\sqrt{\Delta_n}}{2\rho_0}t\right)$$

The rest is the same as 4.4.3.

□

7. Solve ~~the~~ $\begin{cases} \partial_t^2 u - c^2 \partial_x^2 u = 0 \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \quad \partial_t u(x,0) = g(x) = 0 \end{cases}$

Show that $u(x,t) = \frac{1}{2}\left(F(x-ct) + F(x+ct)\right)$

pf: $\partial_t^2 u - c^2 \partial_x^2 u = 0$

$$\Rightarrow (\partial_t + c\partial_x)\underbrace{(\partial_t - c\partial_x) u}_{\text{let } v = \partial_t u - c\partial_x u} = 0.$$

$$\Rightarrow \partial_t v + c\partial_x v = 0$$

$$\Rightarrow v(x,t) = \psi(x-ct) \text{ for } \cancel{\text{some } \psi \text{ with}}\ \psi(x) := v(x,0).$$

then solve $u$ from $v$:

$$\partial_t u - c\partial_x u = \psi(x-ct) \qquad \text{in } \mathbb{R} \times [0,\infty)$$

one can extend $\psi$ from $[-L, L]$ first, then periodically to $\mathbb{R}$.

$$u(x,t) = \int_0^t \psi(\overbrace{x + c(t-s) - cs}^{=y})\, ds + f(x+ct)$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\, dy + f(x+ct)$$

change of variables

Now, observe that $\psi(x) = v(x,0) = \partial_t u(x,0) - c\partial_x u(x,0)$
$= -cf'(x)$

$$\Rightarrow u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} -cf'(y)\, dy + f(x+ct)$$

$$= \frac{1}{2}\left(f(x+ct) + f(x-ct)\right).$$

This is called (D'Alembert formula)

$\square$