Undergrad PDE

HW10

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[7.3] 3. Solve $\partial_t u = k_1 \partial_x^2 u + k_2 \partial_y^2 u$ on a rectangle $(0,L)\times(0,H)$ subject to

$$\begin{cases} u(x,y;0) = f(x,y) \\ u(0,y;t) = u(L,y;t) = 0 \\ \partial_y u(x,0;t) = 0 \quad \partial_y u(x,H;t) = 0 \end{cases}$$

Sketch of proof: Let $u(x,y;t) = X(x)Y(y)T(t)$.

then we get $\frac{T'}{T} = k_1 \frac{X''}{X} + k_2 \frac{Y''}{Y} = -\lambda$.

so $T(t) = Ce^{-\lambda t}$.

The boundary data gives: $X(0) = X(L) = 0$. $Y'(0) = Y'(L) = 0$.

Now we suppose $k_2 \frac{Y''}{Y} + \lambda = -k_1 \frac{X''}{X} =: \mu$

$$\Rightarrow \begin{cases} X'' + \frac{\mu}{k_1} X = 0 \\ X(0) = X(L) = 0 \end{cases} \Rightarrow \begin{cases} X_n(x) = \sin\frac{n\pi x}{L} \\ \mu_n = k_1\left(\frac{n\pi}{L}\right)^2 \quad n \geq 1 \end{cases}$$

Thus,

$$\begin{cases} Y''(y) + \frac{\lambda - \mu_n}{k_2} Y(y) = 0 \\ Y'(0) = Y'(H) = 0 \end{cases} \Rightarrow \begin{cases} Y_m(y) = \cos\frac{m\pi y}{H} \\ \frac{\lambda_{mn} - \mu_n}{k_2} = \left(\frac{m\pi}{H}\right)^2 \quad m \geq 1 \end{cases}$$

$\Rightarrow \lambda_{m,n} = \pi^2\left(k_1 \frac{n^2}{L^2} + k_2 \frac{m^2}{H^2}\right)$ eigenvalues $m,n \geq 1$

eigenfunctions $u_{m,n}(x,y;t) = e^{-\lambda_{m,n} t} \sin\frac{n\pi x}{L} \cos\frac{m\pi y}{H}$ $m,n \geq 1$

$\Rightarrow u(x,y;t) = \sum_{m,n} A_{mn} u_{mn}(x,y;t)$

Finally, using initial data, we know

$$A_{m,n} = \frac{4}{LH}\int_0^L \int_0^H f(x,y) \sin\frac{n\pi x}{L} \cos\frac{m\pi y}{H}\, dy\, dx$$

[7.4] 1 Consider the eigenvalue problem:

$$\begin{cases} \Delta\phi + \lambda\phi = 0 \\ \partial_x\phi(0,y) = \partial_x\phi(L,y) = 0 \\ \phi(x,0) = \phi(x,H) = 0 \end{cases}$$

(1) Show that there is a doubly infinite set of eigenvalues

(2) If $L=H$, show that most eigenvalues have more than one eigenfunction

(3) Derive that the eigenfunctions are orthogonal in a 2D sense using two 1D orthogonality relations.

<u>proof</u>; (1) Assume $\phi(x,y) = X(x)Y(y)$.

$$\Rightarrow X''Y + XY'' + \lambda XY = 0$$

$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} - \lambda = -\mu.$$

with boundary condition $\begin{cases} \partial_x f(0) = 0 = \partial_x f(L) \\ g(0) = 0 = g(H) \end{cases}$

We can solve: $X_n(x) = \cos\left(\frac{n\pi}{L}x\right)$ with $\mu_n = \left(\frac{n\pi}{L}\right)^2$ $n \geq 0$

Then: $\frac{Y''}{Y} = \mu_n - \lambda_{mn}$

$\Rightarrow Y_{mn}(y) = \sin\frac{m\pi}{H}y$ $m \geq 1$ with $\lambda_{mn} - \mu_n = \left(\frac{m\pi}{H}\right)^2$

therefore, $\lambda_{m,n} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$ $n \geq 0$, $m \geq 1$. is a doubly infinite set of eigenvalues

(2) $L = H \Rightarrow \lambda_{m,n} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2$

$\phi_{m,n}(x,y) = \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi}{H}y\right)$ $n \geq 0$, $m \geq 1$,

and $\phi_{n,m}(x,y) = \cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)$ } are both eigenfunctions

when $m \neq n$, they are different

(3). $\int_0^L\int_0^H \phi_{m,n}\phi_{p,q}\,dx\,dy = \int_0^L\int_0^H \cos\frac{n\pi x}{L}\sin\left(\frac{m\pi}{H}y\right)\cos\left(\frac{q\pi x}{L}\right)\sin\left(\frac{p\pi x}{H}\right)dx\,dy$

$= \frac{4}{HL}\,\delta_{mn}\,\delta_{pq}$ (δ_{mn}: 1 iff $m=n$; δ_{pq}: 1 iff $p \neq q$)

$\Rightarrow$ orthogonal. $\square$

[7.5] 9. Show that $\int (u\nabla v - v\nabla u)\cdot\hat{n}\, dS = 0$ if u and v satisfy

(1) Assume $\beta_2 \neq 0$ (7.5.2) $\beta_1\phi + \beta_2\nabla\phi\cdot\hat{n} = 0$

(2) Assume $\beta_2 = 0$ for part of the bdry

proof: (1) $\int (\cancel{u}\nabla v - v\nabla u)\cdot\hat{n}\, dS$

$$= \int u(\nabla v\cdot\hat{n}) - v(\nabla u\cdot\hat{n})\, dS$$

$$= \int u\left(-\frac{\beta_1}{\beta_2}v\right) - v\left(-\frac{\beta_1}{\beta_2}u\right) dS = \int 0\, dS = 0. \quad \beta_2 \neq 0$$

(2) Assume $\beta_2 = 0$ for $C_1 \subset \underset{C_1 \cup C_2}{\underbrace{\text{boundary}}}$

on C_1: $\beta_1\phi + \beta_2\nabla\phi\cdot\hat{n} = \beta_1\phi = 0 \Rightarrow \phi = 0$

$$\int (u\nabla v - v\nabla u)\cdot\hat{n}\, dS$$

$$= \left(\int_{C_1} + \int_{C_2}\right)(u\nabla v - v\nabla u)\cdot\hat{n}\, dS$$

Apply (1) to $C_2 \Rightarrow \int_{C_2} = 0$

~~Apply~~ Then note that $\phi = 0$ on $C_1 \Rightarrow \int_{C_1} = 0$

$\Big\} \Rightarrow \int (u\nabla v - v\nabla u)\cdot\hat{n}\, dS = 0$. $\square$

[7.6] 4. (1) If $\nabla^2\phi = 0$ with $\phi = 0$ on the bdry, prove that $\phi = 0$.

(2). Prove that there cannot be 2 different Soln of the problem

$\nabla^2 u = f(x,y,z)$ subject to bdry data $u = g(x,y,z)$

pf: (1) $$\int_U |\nabla\phi|^2\, dx = -\int_U \phi\cdot\underset{0}{\underbrace{\nabla^2\phi}}\, dx + \int_{\underset{\text{boundary}}{\partial U}} \phi\, \underset{0}{\underbrace{\frac{\partial\phi}{\partial n}}} \equiv 0$$

$\Rightarrow \phi = $ constant

$\phi|_{\partial U} = 0 \Rightarrow \phi \equiv 0$.

(2). If u_1, u_2 are solutions, then $u = u_1 - u_2$ satisfies

$$\begin{cases} \nabla^2 u = 0 & \text{in } U \\ u|_{\partial U} = 0 \end{cases} \Rightarrow u = 0. \quad \square$$

[7.7] 10. Solve $u(r,t)$ if it satisfies the circularly symmetric heat equ. for $\partial_t u = \frac{k}{r}\partial_r(r\,\partial_r u)$. subject to the conditions $\begin{cases} u(a,t)=0 \\ u(r,0)=f(r) \end{cases}$

Briefly analyze $\lim_{t\to\infty}$

Sol: Let $u(r,t) = R(r)\,T(t) \Rightarrow R(r)\,T'(t) = \frac{k}{r}\partial_r(r\,R'(r)T(t))$

$$\Rightarrow \frac{1}{k}\frac{T'(t)}{T(t)} = \frac{1}{r}\partial_r\left(r\,R'(r)\right) = -\lambda$$

For $T(t)$, we have $T'(t) + \lambda k T(t) = 0 \Rightarrow T(t) = T(0)\,e^{-\lambda k t}$.

For $R(r)$: $\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda r R = 0$

$$\Rightarrow \frac{dR}{dr} + r\frac{d^2R}{dr^2} + \lambda r R = 0 \Rightarrow r^2 R''(r) + rR'(r) + \lambda r^2 R(r) = 0$$

Let $z = \sqrt{\lambda}\,r \Rightarrow z^2 R''(z) + zR'(z) + z^2R(z) = 0$

$$\Rightarrow \cancel{\phi}\, R(r) = C_1 J_0(\sqrt{\lambda}\,r) + C_2 Y_0(\sqrt{\lambda}\,r)$$

Since $|\phi(0)| < \infty \Rightarrow C_2 = 0$

$$\Rightarrow R(r) = C_1 J_0(\sqrt{\lambda}\,r)$$

Boundary data $\Rightarrow J_0(\sqrt{\lambda}\,a) = 0$

So: $u(r,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n k t} J_0(\sqrt{\lambda_n}\,r)$

$u(r,0) = f(r) = \sum_{n=1}^{\infty} a_n J_0(\sqrt{\lambda_n}\,r)$

which gives $a_n = \dfrac{\int_0^a f(r)\,J_0(\sqrt{\lambda_n}\,r)\,r\,dr}{\int_0^a J_0^2(\sqrt{\lambda_n}\,r)\,r\,dr}$

As $t\to\infty$, $u \to 0$ b/c $e^{-\lambda_n k t}$ and J_0 are both decaying terms. $\square$

12(b) What is the expected approximate behaviour of all solns near $x=0$

$$x^2 y'' + (x^2 + \tfrac{3}{16})y = 0$$

Near $x=0$ $\quad x^2 y'' + \frac{3}{16} y \sim 0.$

$$\Rightarrow \quad y \sim C_1 x^{\frac{3}{4}} + C_2 x^{\frac{1}{4}}$$ (Solve this homogeneous ODE).

$\square$