Chapter 8: Non-homogeneous Equations

Fei Lu

Department of Mathematics, Johns Hopkins

Solution to the IBVP?

$$\partial_t u = \kappa \partial_{xx} u + Q(x,t), \quad \text{with } x \in (0,L), t \geq 0$$
$$u(x,0) = f(x)$$
$$\text{BC: } u(0,t) = \phi(t), u(L,t) = \psi(t)$$

Section 8.2: Heat flow with source and non-homo BC
Section 8.3: Methods of eigenfunction expansion (homo-BC)
Section 8.4: MEE (non-homo BC): after Chp5
Section 8.5: Forced vibrating membrane and Resonance
Section 8.6: Poisson's Equition

Outline

Section 8.2: Heat flow with source and non-homo BC

1. Time-independent BC

Consider first

$\partial_t u = \kappa \partial_{xx} u,$ with $x \in (0, L), t > 0$

$u(x, 0) = f(x)$

BC: $u(0, t) = A, u(L, t) = B$

1> Equilibrium solu $u_E(x) = A + \frac{x}{L}(B - A)$.

Section 8.2: Heat flow with source and non-homo BC

1. Time-independent BC

Consider first

$$\partial_t u = \kappa \partial_{xx} u, \quad \text{with } x \in (0, L), t > 0$$
$$u(x,0) = f(x)$$
$$\text{BC: } u(0,t) = A, u(L,t) = B$$

1> Equilibrium solu $u_E(x) = A + \frac{x}{L}(B - A)$.

2> Displacement from Equilibrium

$$v(x,t) = u(x,t) - u_E(x)$$

$$\partial_t v = \kappa \partial_{xx} v,$$
$$v(x,0) = f(x) - u_E(x)$$
$$v(0,t) = 0, v(L,t) = 0$$

$$v(x,t) = \sum_{n=0}^{\infty} a_n e^{-\kappa \lambda_n t} \sin \frac{n\pi}{L} x$$

Section 8.2: Heat flow with source and non-homo BC

1. Time-independent BC

Consider first

$$\partial_t u = \kappa \partial_{xx} u, \quad \text{with } x \in (0,L), t > 0$$
$$u(x,0) = f(x)$$
$$\text{BC: } u(0,t) = A, u(L,t) = B$$

1> Equilibrium solu $u_E(x) = A + \frac{x}{L}(B - A)$.

2> Displacement from Equilibrium

$$v(x,t) = u(x,t) - u_E(x)$$

$$\partial_t v = \kappa \partial_{xx} v,$$
$$v(x,0) = f(x) - u_E(x)$$
$$v(0,t) = 0, v(L,t) = 0$$

$$v(x,t) = \sum_{n=0}^{\infty} a_n e^{-\kappa \lambda_n t} \sin \frac{n\pi}{L} x$$

Extension (exe): steady source

$$\partial_t u = \kappa \partial_{xx} u + Q(x)$$
$$u(x,0) = f(x)$$
$$u(0,t) = A, u(L,t) = B$$

2. Time-dependent non-homo PDE&BC

$$\partial_t u = \kappa \partial_{xx} u + Q(x,t),$$
$$u(x,0) = f(x)$$
$$u(0,t) = A(t), u(L,t) = B(t)$$

1> Homogenization:

- Equilibrium solu?
- May NOT be able o reduce both PDE and BC to homo.
  choose one: PDE or BC?

2. Time-dependent non-homo PDE&BC

$$\partial_t u = \kappa \partial_{xx} u + Q(x,t),$$
$$u(x,0) = f(x)$$
$$u(0,t) = A(t), u(L,t) = B(t)$$

1> Homogenization:

- Equilibrium solu?
- May NOT be able o reduce both PDE and BC to homo.
  choose one: PDE or BC?

$\Rightarrow$ reference solution $r(x,t)$ s.t.

$$r(0,t) = A(t); r(L,t) = B(t)$$

- any r ***
- $r(x,t) = A(t) + \frac{x}{L}[B(t) - A(t)]$.

2. Time-dependent non-homo PDE&BC

$$\partial_t u = \kappa \partial_{xx} u + Q(x,t),$$
$$u(x,0) = f(x)$$
$$u(0,t) = A(t), u(L,t) = B(t)$$

1> Homogenization:

- Equilibrium solu?
- May NOT be able o reduce both PDE and BC to homo.
  choose one: PDE or BC?

$\Rightarrow$ reference solution $r(x,t)$ s.t.

$$r(0,t) = A(t); r(L,t) = B(t)$$

- any r ***
- $r(x,t) = A(t) + \frac{x}{L}[B(t) - A(t)]$.

2> Displacement solution
$v(x,t) = u(x,t) - r(x,t)$

$$\partial_t v = \kappa \partial_{xx} v + \overline{Q}(x,t),$$
$$v(x,0) = f(x) - r(x,0)$$
$$v(0,t) = 0, v(L,t) = 0$$

- $\overline{Q} = ?$: ***
- can we use separation of variables?
  $v(x,t) = h(t)\phi(x)$
  $\overline{Q}$ and no POS.
- Fourier series
  $v(x,t) \text{“}=\text{”} \sum_{n=0}^{\infty} a_n(t) \sin \frac{n\pi}{L} x$
  $\overline{Q}(x,t) \text{“}=\text{”} \sum_{n=0}^{\infty} q_n(t) \sin \frac{n\pi}{L} x$
  $\rightarrow$ method of eigenfunction expansion $\downarrow$

Outline

Section 8.3: Methods of eigenfunction expansion

separation of variables: homo PDE + homo BC

generalize $\to$ non-homo PDE + homogeneous BC
Seek solution of the form

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos \frac{n\pi}{L} x + b_n(t) \sin \frac{n\pi}{L} x,$$

- homo BC determines the eigenfunctions to use (sine/cosine/both, denote by $\phi_n(x)$)
- works for equation with source $\partial_t u = \kappa \partial_{xx} u + Q(x,t)$
- solve $a_n(t), b_n(t)$ from the PDE + IC (use TBTD, under conditions)

Method of eigenfunction expansion

$$\partial_t v = \kappa \partial_{xx} v + \overline{Q}(x,t),$$
$$v(x,0) = g(x)$$
$$v(0,t) = 0, v(L,t) = 0$$

$$\phi_n(x) = \sin \frac{n\pi}{L} x$$

$$v(x,t) \text{``}=\text{''} \sum_{n=0}^{\infty} b_n(t)\phi_n(x), \quad (b_n \text{ TBD } \downarrow)$$

$$g(x) \text{``}=\text{''} \sum_{n=0}^{\infty} b_n(0)\phi_n(x), \quad b_n(0) = ?$$

$$\overline{Q}(x,t) \text{``}=\text{''} \sum_{n=0}^{\infty} \overline{q_n}(t)\phi_n(x)$$

Method of eigenfunction expansion

$$\partial_t v = \kappa \partial_{xx} v + \overline{Q}(x,t),$$
$$v(x,0) = g(x)$$
$$v(0,t) = 0, v(L,t) = 0$$

$$\phi_n(x) = \sin \tfrac{n\pi}{L} x$$

$$v(x,t) \text{“}=\text{”} \sum_{n=0}^{\infty} b_n(t)\phi_n(x), \quad (b_n \text{ TBD } \downarrow)$$

$$g(x) \text{“}=\text{”} \sum_{n=0}^{\infty} b_n(0)\phi_n(x), \quad b_n(0) = ?$$

$$\overline{Q}(x,t) \text{“}=\text{”} \sum_{n=0}^{\infty} \overline{q_n}(t)\phi_n(x)$$

TBTD $\partial_t v, \partial_{xx} v$ PS; $v, \partial_x v$ continuous; (BC?)$\Rightarrow$

- $\partial_t v = \sum_{n=0}^{\infty} b_n'(t)\phi_n(x)$
  $\kappa \partial_{xx} v + \overline{Q}(x,t) = \sum_{n=0}^{\infty} [-\lambda_n \kappa b_n(t) + \overline{q_n}(t)]\phi_n(x)$

  $$\Rightarrow b_n' + \lambda_n \kappa b_n(t) = \overline{q_n}(t)$$

  $$b_n(t) = b_n(0)e^{-\kappa\lambda_n t} + \int_0^t e^{-\kappa\lambda_n(t-s)}\overline{q_n}(s)ds$$

- Check: if $\overline{Q}(x,t) = 0$: $b_n(t) = b_n(0)e^{-\kappa\lambda_n t}$.

Example

Find a solution of

$$\partial_t u = \kappa \partial_{xx} u + e^{-t} \sin 3x$$
$$u(x,0) = f(x)$$
$$u(0,t) = 0, u(\pi,t) = 1$$

Example

Find a solution of

$$\partial_t u = \kappa \partial_{xx} u + e^{-t} \sin 3x$$
$$u(x,0) = f(x)$$
$$u(0,t) = 0, u(\pi,t) = 1$$

$L = \pi$, $\lambda_n = n^2$;

1> reference solution:

$r(x,t) = 0 + \frac{x}{\pi}(1-0) = \frac{x}{\pi}$

Example

Find a solution of

$$\partial_t u = \kappa \partial_{xx} u + e^{-t} \sin 3x$$
$$u(x,0) = f(x)$$
$$u(0,t) = 0, u(\pi,t) = 1$$

$L = \pi$, $\lambda_n = n^2$;

1> reference solution:

$r(x,t) = 0 + \frac{x}{\pi}(1-0) = \frac{x}{\pi}$

2> let $v(x,t) = u(x,t) - r(x,t)$

$$\partial_t v = \kappa \partial_{xx} v + \overline{Q}(x,t),$$
$$v(x,0) = f(x) - r(x,0)$$
$$v(0,t) = 0, v(L,t) = 0$$

- $\overline{Q}(x,t) = e^{-t} \sin 3x + 0$
- BC$\Rightarrow$ $\phi_n(x) = \sin nx$

Seek $v(x,t) = \sum_{n=0}^{\infty} b_n(t) \phi_n(x)$

Example

Find a solution of

$$\partial_t u = \kappa \partial_{xx} u + e^{-t} \sin 3x$$
$$u(x,0) = f(x)$$
$$u(0,t) = 0, u(\pi,t) = 1$$

$L = \pi$, $\lambda_n = n^2$;

1> reference solution:

$r(x,t) = 0 + \frac{x}{\pi}(1-0) = \frac{x}{\pi}$

2> let $v(x,t) = u(x,t) - r(x,t)$

$$\partial_t v = \kappa \partial_{xx} v + \overline{Q}(x,t),$$
$$v(x,0) = f(x) - r(x,0)$$
$$v(0,t) = 0, v(L,t) = 0$$

- $\overline{Q}(x,t) = e^{-t} \sin 3x + 0$
- BC$\Rightarrow$ $\phi_n(x) = \sin nx$

Seek $v(x,t) = \sum_{n=0}^{\infty} b_n(t) \phi_n(x)$

TBTD $\partial_t v, \partial_{xx} v$ PS; $v, \partial_x v$ continuous; $\Rightarrow$

- $b_n(t) = b_n(0) e^{-\kappa \lambda_n t} + \int_0^t e^{-\kappa \lambda_n (t-s)} \overline{q_n}(s) ds$

$u(x,t) = v(x,t) + \frac{x}{\pi}$

Outline

In the method of eigenfunction expansion, what if

- TBTD conditions not satisfied
- More general equations $\partial_{xx} \to \frac{d}{dx}(p(x)\frac{d}{dx})$

Back to it after chapter 5.

Outline

Outline