Chapter 2: Method of Separation of Variables

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Solution to the IBVP?

$$\partial_t u = \kappa \partial_{xx} u + Q(x,t), \quad \text{with } x \in (0,L), t \geq 0$$
$$u(x,0) = f(x)$$
$$\text{BC: } u(0,t) = \phi(t), u(L,t) = \psi(t)$$

Solution to the IBVP?

$$\begin{aligned}
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&u(x,0) = f(x) \\
&u(0,t) = \phi(t), u(L,t) = \psi(t)
\end{aligned}$$

Recall ODEs:

$$\underbrace{ay'' + by' + cy}_{Ly} = g(x); \quad y(x_0) = \alpha; y(x_1) = \beta.$$

- Step 1: solve the **linear** equation $Ly = 0 \Rightarrow y_1(x), y_2(x)$
- Step 2: find the specific solution $Ly = g \Rightarrow y_s(x)$

$\Rightarrow$ general solution: $y = c_1 y_1 + c_2 y_2 + y_s$ with c_1, c_2 TBD by BC/IC.

Same for PDE? key principles?

linear homogeneous $\Rightarrow$ Principle of Superposition (PoS)

Outline

Section 2.2: Linearity

Section 2.3: HE with zero boundaries

Section 2.4: HE with other boundary values

Section 2.2: Linearity

Linear operator: for any $c_1, c_2 \in \mathbb{R}$,

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2), \quad \forall u_1, u_2 \in Dom(L)$$

Examples: which operator(s) nonlinear?

A. $L = \partial_{xxx}$;

B. $L = \partial_t - \kappa \partial_{xx}$;

C. $L(u) = \partial_x (K(x) \partial_x u)$;

D. $L(u) = \partial_{xx} u + u \partial_x u$

E. $L(u) = u(x, 0)$

F. $L(u) = c_1 u(0, t) + c_2 \partial_x u(1, t)$

Linear homogeneous equation $L(u) = f$ with $f = 0$
otherwise (if $f \neq 0$), nonhomogeneous.

- linearity and homogeneity also apply to BC.

Principle of Superposition L linear,

$$\text{if } L(u_1) = L(u_2) = 0, \text{ then } L(c_1 u_1 + c_2 u_2) = 0 \text{ .}$$

- if u_1, u_2 solve $L(u) = 0$, then so does $c_1 u_1 + c_2 u_2$
- T/F? $L(u_1) = f_1, L(u_2) = f_2 \Rightarrow L(u_1 + u_2) = f_1 + f_2$.

??? Is $u = v + w$ a solution to

$$\begin{aligned} \partial_t u = &\ \kappa \partial_{xx} u + Q(x,t), \qquad \text{with } x \in (0,1), t \geq 0 \\ u(x,0) &= f(x) \\ u(0,t) &= \phi(t), u(1,t) = \psi(t) \end{aligned}$$

if

$$\begin{aligned} \partial_t v = &\ \kappa \partial_{xx} v, \\ v(x,0) &= f(x) \\ v(0,t) &= 0, v(1,t) = 0 \end{aligned}$$

$$\begin{aligned} \partial_t w = &\ \kappa \partial_{xx} w + Q(x,t), \\ w(x,0) &= 0 \\ w(0,t) &= \phi(t), u(1,t) = \psi(t) \end{aligned}$$

Outline

HE: homogeneous IBVP

$$\partial_t u = \kappa \partial_{xx} u,$$
$$u(x,0) = f(x)$$
$$u(0,t) = 0, u(L,t) = 0$$

- equation and BC: linear homogeneous
- physical meaning:
  1D rod with no sources and both ends immersed at 0^o.
  How the temperature evolve to Equilibrium?
- a first step for general IBVP (from previous slide)
  can be solved by **method of separation of variables** $\downarrow$

Separation of variables

Seek solutions in the form (Daniel Bernoulli 1700s)

$$u(x,t) = \phi(x)G(t)$$

Reduce PDE to ODEs:

$$\partial_t u = \phi(x)G'(t) = \kappa \partial_{xx} u = \kappa \phi''(x)G(t)$$

$$\frac{G'(t)}{\kappa G(t)} = \frac{\phi''(x)}{\phi(x)} \overset{\text{for any x,t}}{=} -\lambda$$

- λ is a constant TBD
- two ODEs:
  In time: $G'(t) = -\lambda\kappa G(t) \quad \Rightarrow$
  In space: $\phi''(x) = -\lambda\phi(x) \quad \Rightarrow$
- IC: trivial solution when $f(x) = 0$, $u \equiv 0$ with $G \equiv 0$;
  otherwise, $u(x,0) = G(0)\phi(x) = f(x)$: $G(0)$ TBD
- BC: for non-trivial solution $\Rightarrow \phi(0) = \phi(L) = 0$

Time dependent ODE

$$G'(t) = -\lambda\kappa G(t) \quad \Rightarrow G(t) = G(0)e^{-\lambda\kappa t}.$$

Assume that $G(0) > 0$,

- $\lambda < 0$: $G(t) \uparrow \infty$
- $\lambda = 0$:
- $\lambda > 0$:

Physical setting: $\lambda \geq 0$

Boundary value problem

$$\phi''(x) = -\lambda\phi(x), \quad \phi(0) = \phi(L) = 0$$

- $\lambda < 0$: $\phi(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$
- $\lambda = 0$: $\phi(x) =$
- $\lambda > 0$: $\phi(x) =$

Eigenfunctions: $\mathbf{L}\phi = \lambda\phi, \phi(0) = \phi(L) = 0$, with $\mathbf{L}\phi := -\phi''$

$$\phi_n(x) = \sin(\frac{n\pi}{L}x), \quad \lambda_n = (\frac{n\pi}{L})^2, \quad n = 1, 2, \cdots,$$

Solution to HE-IBVP:

$$\partial_t u = \kappa \partial_{xx} u,$$
$$u(x,0) = f(x)$$
$$u(0,t) = 0, u(L,t) = 0$$

$$\lambda_n = (\tfrac{n\pi}{L})^2, n = 1, 2, \ldots$$

$$u(x,t) = \phi_n(x) G_n(t) = \sin(\frac{n\pi}{L}x) e^{-\lambda_n \kappa t}$$

PoS:

$$u_N(x,t) = \sum_{n=1}^{N} B_n \sin(\frac{n\pi}{L}x) e^{-\lambda_n \kappa t} \rightarrow u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{L}x) e^{-\lambda_n \kappa t}$$

- if $f(x) = \sum_{n=1}^{N} B_n \sin(\frac{n\pi}{L}x)$, u_N is a solution
- if $f(x) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{L}x)$, u is a solution
  (convergence of function series: Chp3:Fourier series)

For a general f, how to determine B_n? **Orthogonality**

$$\int_0^L \sin(\frac{n\pi}{L}x) \sin(\frac{m\pi}{L}x) dx = \delta_{m-n} \frac{L}{2}$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi}{L}x) dx$$

2.3.8 Summary

Let us summarize the method of separation of variables as it appears for the one example:

$$\begin{aligned} \text{PDE:} \quad & \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \\ \text{BC:} \quad & u(0,t) = 0 \\ & u(L,t) = 0 \\ \text{IC:} \quad & u(x,0) = f(x). \end{aligned}$$

1. Make sure that you have a linear and homogeneous PDE with linear and homogeneous BC.
2. Temporarily ignore the nonzero IC.
3. Separate variables (determine differential equations implied by the assumption of product solutions) and introduce a separation constant.
4. Determine separation constants as the eigenvalues of a boundary value problem.
5. Solve other differential equations. Record all product solutions of the PDE obtainable by this method.
6. Apply the principle of superposition (for a linear combination of all product solutions).
7. Attempt to satisfy the initial condition.
8. Determine coefficients using the orthogonality of the eigenfunctions.

These steps should be *understood*, not memorized. It is important to note that

1. The principle of superposition applies to solutions of the PDE (do not add up solutions of various different ordinary differential equations).
2. Do not apply the initial condition $u(x,0) = f(x)$ until *after* the principle of superposition.

Outline

Section 2.4: HE with other boundary values

$$\partial_t u = \kappa \partial_{xx} u,$$
$$u(x,0) = f(x)$$
$$\partial_x u(0,t) = 0, \partial_x u(L,t) = 0$$

$$\lambda_n = (\tfrac{n\pi}{L})^2, n = 1, 2, \ldots$$

$$u(x,t) = \phi_n(x) G_n(t) = \cos(\frac{n\pi}{L} x) e^{-\lambda_n \kappa t}$$

Review of the method: separation of variables (SoV)

$$\underbrace{PDE}_{\text{linear, homo}} + \underbrace{BC}_{\text{linear, homo}} + IC$$

$$\partial_t u = \kappa \partial_{xx} u,$$
$$u(0,t) = 0, u(L,t) = 0$$
$$u(x,0) = f(x)$$

1. linear + homo $\Rightarrow$ PoS

2. SoV: PDE+BC $\Rightarrow$ ODEs

$$\frac{G'(t)}{\kappa G(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda$$

3. Solve EigenvalueP

$$G(t) = G(0)e^{-\lambda\kappa t}.$$

$$\phi''(x) = -\lambda\phi(x), \quad \phi(0) = \phi(L) = 0$$

4. IC $\Rightarrow$ coefficients

(orthogonality $\downarrow$)

$$\phi_n(x) = \sin(\frac{n\pi}{L}x),\ \lambda_n = (\frac{n\pi}{L})^2, n \geq 1$$

5. Conclude solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{L}x)e^{-\lambda_n \kappa t}$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx$$

Orthogonality

In finite dimensional space: $\mathbf{a} = (a_1, a_2, \ldots, a_N), \mathbf{b} \in \mathbb{R}^N$:

$$\mathbf{a} \perp \mathbf{b} \Leftrightarrow \langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{N} a_i b_i = 0$$

For functions: $\phi, \psi \in C[0, L]$ (connection?)

$$\phi \perp \psi \Leftrightarrow \langle \phi, \psi \rangle = \int_0^L \phi(x)\psi(x)dx = 0$$

Recall $\{\phi_n, \lambda_n\}$ with $\phi_n(x) = \sin(\frac{n\pi}{L}x)$ and $\lambda_n = \frac{n\pi}{L}$ solve:

$$\phi''(x) = -\lambda\phi(x), \quad \phi(0) = \phi(L) = 0$$

We have $\langle \phi_n, \phi_m \rangle = \delta_{m-n}\frac{L}{2}$.

HE+ $BC_{\text{Neumann, homo}}$ + IC

$$\partial_t u = \kappa \partial_{xx} u,$$
$$\partial_x u(0,t) = 0, \partial_x u(L,t) = 0$$
$$u(x,0) = f(x)$$

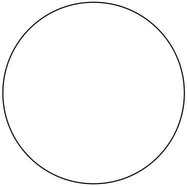
1. linear homo: ⇒ PoS
2. SoV: $u(x,t) = \phi(x)G(t)$
3. Solve EigenvalueP
4. Determine coefs. by IC.
5. Conclude solution

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda_n \kappa t} \phi_n(x)$$

$\lim_{t\to\infty} u(x,t) =?$

HE in a circular ring

$$\begin{aligned}
\partial_t u &= \kappa \partial_{xx} u, \\
u(L,t) &= u(-L,t) \\
\partial_x u(L,t) &= \partial_x u(-L,t) \\
u(x,0) &= f(x)
\end{aligned}$$



1. linear homo: $\Rightarrow$ PoS
2. SoV: $u(x,t) = \phi(x)G(t)$
3. Solve EigenvalueP
4. Determine coefs. by IC.
5. Conclude solution

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} e^{-\lambda_n \kappa t}[a_n \phi_n(x) + b_n \psi_n(x)]$$

$\lim_{t\to\infty} u(x,t) = ?$

Summary of boundary value problems for $\phi'' = -\lambda\phi$:

BOUNDARY VALUE PROBLEMS

Boundary conditions	$\phi(0) = 0$ $\phi(L) = 0$	$\frac{d\phi}{dx}(0) = 0$ $\frac{d\phi}{dx}(L) = 0$	$\phi(-L) = \phi(L)$ $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$
Eigenvalues λ_n	$\left(\frac{n\pi}{L}\right)^2$ $n = 1,\ 2,\ 3, \ldots$	$\left(\frac{n\pi}{L}\right)^2$ $n = 0,\ 1,\ 2,\ 3, \ldots$	$\left(\frac{n\pi}{L}\right)^2$ $n = 0,\ 1,\ 2,\ 3, \ldots$
Eigenfunctions	$\sin \frac{n\pi x}{L}$	$\cos \frac{n\pi x}{L}$	$\sin \frac{n\pi x}{L}$ and $\cos \frac{n\pi x}{L}$
Series	$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$
Coefficients	$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$	$A_0 = \frac{1}{L} \int_0^L f(x)\, dx$ $A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx$	$a_0 = \frac{1}{2L} \int_{-L}^L f(x)\, dx$ $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L}\, dx$ $b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L}\, dx$