Chap 19: 12. 15. 16. 19. 24. 33. 39. 40

12. We say that (f_n) is Cauchy in measure if, given $\varepsilon>0$, there exists an N s.t $m\{|f_n-f_m|\geq\varepsilon\}<\varepsilon$ whenever $m,n>N$. If (f_n) converges in measure, show that (f_n) is necessarily Cauchy in measure

pf: Since (f_n) converges in measure to f, then

$\forall\ \varepsilon>0,\ \exists\ N$ s.t $\quad m\{|f_n-f|\geq\frac{\varepsilon}{2}\}<\frac{\varepsilon}{2}\quad$ for all $n>N$

Furthermore, we claim:

$$\{|f_n-f_m|\geq\varepsilon\}\subset\{|f_n-f|\geq\tfrac{\varepsilon}{2}\}\cup\{|f_m-f|\geq\tfrac{\varepsilon}{2}\}$$

otherwise, $|f_n-f_m|\leq|f_n-f|+|f_m-f|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

Therefore, for $n,m>N$

$$m\{|f_n-f_m|\geq\varepsilon\}\leq m\{|f_n-f|\geq\tfrac{\varepsilon}{2}\}+m\{|f_m-f|\geq\tfrac{\varepsilon}{2}\}<\varepsilon$$

15 If $f_n\to f$ in L_1, prove that there is a subseq of (f_n) that converges almost uniformly to f.

pf: By cor 19.5. we can find a subseq $(f_{n_k})\xrightarrow{a.e}f$. Then, we can choose a further subseq g_{n_k} s.t $\int|g_{n_k}-f|<2^{-n}$ for all n.

Denote g_{n_k} by f_n without misunderstanding.

Now, we repeat the proof of Egorov's Theorem (Thm 17.13)

Consider $\quad E(1,k)=\bigcup_{n=1}^{\infty}\{x\in D:|f_n(x)-f(x)|\geq\frac{1}{k}\}$

Denote $\quad E_{k,n}=\{x\in D:|f_n(x)-f(x)|\geq\frac{1}{k}\}$.

Then $\quad m(E_{k,n})\cdot\frac{1}{k}\leq\int|f_n-f|<2^{-n}\quad\Rightarrow m(E_{k,n})<2^{-n}\cdot k$

$$\Rightarrow\ m(E(1,k))<\sum_{n=1}^{\infty}2^{-n}\cdot k=k<+\infty.$$

Then we can repeat the remaining part of the proof of Egorov's Theorem

$\boxed{16}$ Over a set of finite measure, we can actually describe convergence in measure in terms of a metric. For example, consider

$$d(f,g) = \int_0^1 \min\{|f(x)-g(x)|, 1\}\,dx$$

where f, g : measurable, real-valued functions on $[0,1]$

(a) check that $d(f,g)$ is pseudometric, with $d(f,g)=0$ iff $f=g$ a.e

(b) prove that (f_n) converges in measure to f on $[0,1]$ iff $d(f_n,f)\to 0$ as $n\to\infty$

(c) prove that (f_n) is d-cauchy iff (f_n) is Cauchy in measure

pf: (a) 1) $d(f,f)=0$ ✓

2) $d(f,g)=d(g,f)$ ✓

3) claim: $\min\{|x-y|,1\} \le \min\{|x-z|,1\} + \min\{|z-y|,1\}$

if $|x-z|\ge 1$ or $|z-y|\ge 1$ $\Rightarrow$ $\min\{|x-y|,1\} \le 1 \le \min\{|x-z|,1\} + \min\{|z-y|,1\}$

if $|x-z|\le 1$ and $|y-z|\le 1$ $\Rightarrow$ $\min\{|x-y|,1\} \le |x-y| \le |x-z| + |z-y| = \min\{|x-z|,1\} + \min\{|y-z|,1\}$

Hence,

$d(f,g) \le d(f,h) + d(h,g)$

4) $d(f,g)=0 \Leftrightarrow \min\{|f-g|(x),1\}=0$ a.e $\Leftrightarrow$ $f=g$ a.e.

(b). (only if): Since (f_n) converges in measure to f, $\forall \varepsilon$, $\exists N$, s.t

$m\{x\in D : |f_n(x)-f(x)|\ge\varepsilon\} < \varepsilon$ for all $n>N$

Then $d(f_n,f) = \int_0^1 \min\{|f_n(x)-f(x)|, 1\}\,dx$

$\le m([0,1])\cdot\varepsilon + 1\cdot m\{x\in D : |f_n(x)-f(x)|>\varepsilon\} < 2\varepsilon$.

(if): $\forall \varepsilon<1$, $\exists N$, for $\forall n>N$, we have

$d(f_n,f) = \int_0^1 \min\{|f_n(x)-f(x)|, 1\}\,dx < \varepsilon^2$

Then $m\{x\in D : |f_n(x)-f(x)|\ge\varepsilon\}\cdot\varepsilon \le d(f_n,f) < \varepsilon^2$

$\Rightarrow$ $m\{x\in D : |f_n(x)-f(x)|\ge\varepsilon\} < \varepsilon$

(c) only if: if (f_n) is d Cauchy, i.e

$\forall \varepsilon < 1$, $\exists N$ s.t $\forall n, m > N$, we have

$d(f_n, f_m) = \int_0^1 \min\{|f_n - f_m|, 1\} dx < \varepsilon^2$

$\Rightarrow m\{x \in D : |f_n - f_m| \geq \varepsilon\} \cdot \varepsilon \leq d(f_n, f_m) < \varepsilon^2 \Rightarrow m\{x \in D : |f_n - f_m| \geq \varepsilon\} < \varepsilon$

(if): if (f_n) is Cauchy in measure.

$\forall \varepsilon > 0$. $\exists N$ s.t $\forall n, m > N$. we have

$m\{x \in D : |f_n - f_m| \geq \varepsilon\} < \varepsilon$

$\Rightarrow \int_0^1 \min\{|f_n - f_m|, 1\} dx < \varepsilon \cdot 1 + m([0,1]) \cdot \varepsilon = 2\varepsilon$

$\Rightarrow d(f_n, f_m) \to 0$

19 : In sharp contrast to convergence in measure, the topology of convergence pointwise a.e can not, in general, be described by a metric. To see this, prove that

(a) There is a sequence of meas functions (f_n) on $[0,1]$ that fails to converge pointwise a.e to 0, but such that every subseq of (f_n) has a further subs that does converge pointwise a.e. to 0.

(b) There is no metric ρ on $L_0[0,1]$ satisfying $\rho(f_n, f) \to 0$ iff $f_n \to f$ a.e.

pf - (a). For any $n \in \mathbb{N}^+$, $\exists !\, k, j$ s.t $n = 2^k + j$ and $0 \leq j < 2^k$

Define $A_n = [\frac{j}{2^k}, \frac{j+1}{2^k}]$. $f_n(x) = \chi_{A_n}$

① for $x \in [\frac{j}{2^k}, \frac{j+1}{2^k})$ then $f_n(x) = 1$ for $n = 2^k + j$. but $f_{n+1}(x) = 0$

So (f_n) fails to converge pointwise a.e to 0

② But every subseq of (f_n) has a further sub that does converge pointwise a.e to 0

Since $m(\{|f_n| \geq \varepsilon\}) \to 0$ $(m(A_n) \to 0)$. By Exercise 17

$\Rightarrow$ every subseq of (f_n) has a further subseq that converges pointwise a.e to 0

(b) Suppose a metric. ρ existed on $L_0[0,1]$ such that $\rho(f_n, f) \to 0$

iff $f_n \to f$ pointwise almost everywhere. Then

(f_n) converges to zero a.e $\Leftarrow$ each subseq has a further convergent subseq

But by (a). contradiction !

We conclude that ~~there~~ no such metric ρ can exist

[24] Show that equality holds in Hölder's ineq iff $A|f|^{p-1} = B|g|$ for some nonnegative constants A and B, not both zero, if and only if $C|f|^p = D|g|^q$ for some nonnegative constants C and D, not both zero

pf: In proof of 19.7., Suppose $\|f\|_p > 0$ and $\|g\|_q = 0$ (o.w it's obvious)

The equality is achieved in Young's ineq iff $a^p = b^q$

$\Rightarrow$ The equality is achieved in Hölder's ineq iff $\dfrac{|f(x)|^p}{\|f\|_p^p} = \dfrac{|g(x)|^q}{\|g\|_q^q}$

i.e $C|f|^p = D|g|^q \Leftrightarrow A|f|^{p-1} = B|g|$.

[33] If f and g are disjointly supported elements of L_p. that is, if $fg = 0$ a.e. show $\|f+g\|_p^p = \|f\|_p^p + \|g\|_p^p$.

Pf: Since $fg = 0$ a.e, then $|f+g|^p = |f|^p + |g|^p$ a.e

i.e $\|f+g\|_p^p = \|f\|_p^p + \|g\|_p^p$

$\boxed{39}$ Let $f, f_n \in L_p$, $1 \le p < \infty$, and suppose that $f_n \to f$ a.e show that
$\|f_n - f\|_p \to 0$ iff $\|f_n\|_p \to \|f\|_p$

pf: $\boxed{\text{only if}}$:

since $\|f_n - f\|_p \to 0$, then. $\big| \|f_n\|_p - \|f\|_p \big| \le \|f_n - f\|_p$ by Minkowsk's ineq

$\Rightarrow \|f_n\|_p \to \|f\|_p$

$\boxed{\text{if}}$: Defn $g_n(x) = 2^p(|f_n|^p + |f|^p) - |f_n - f|^p \ge 0$ $\Rightarrow g_n \to 2^{p+1}|f|^p$ a.e

By Fatou's Lemma.

$$2^{p+1}\int |f|^p \le \liminf \int 2^p(|f_n|^p + |f|^p) - |f_n - f|^p$$

$$\le \limsup \int 2^p(|f_n|^p + |f|^p) - \limsup \int |f_n - f|^p$$

since $\|f_n\|_p \to \|f\|_p$

$$= 2^{p+1}\int |f|^p - \limsup \int |f_n - f|^p$$

Hence, $0 \le \liminf \int |f_n - f|^p \le \limsup \int |f_n - f|^p \le 0$

$\Rightarrow \lim\limits_{n\to\infty} \|f_n - f\|_p \to 0$

$\boxed{40}$ For $1 < p < \infty$ and $a, b \ge 0$, show that $a^p + b^p \le (a+b)^p \le 2^{p-1}(a^p + b^p)$

and the reverse inequalitres hold when $0 < p < 1$.

pf: For $1 < p < \infty$, if $a = 0$ or $b = 0$, the inequality holds.

Assume $a \ge b > 0$ the above inequality is equivalent to

$$1 + \left(\frac{b}{a}\right)^p \le \left(1 + \frac{b}{a}\right)^p \le 2^{p-1}\left(1 + \left(\frac{b}{a}\right)^p\right)$$

Define $\varphi(x) = \dfrac{(1+x)^p}{1+x^p}$, $0 \le x \le 1$

Then $$\varphi'(x) = \frac{p(1+x)^{p-1}(1+x^p) - (1+x)^p \cdot p x^{p-1}}{(1+x^p)^2} = \frac{p(1+x)^{p-1}(1 + x^p - (1+x)\cdot x^{p-1})}{(1+x^p)^2}$$

$$= \frac{p(1+x)^{p-1}(1 - x^{p-1})}{(1+x^p)^2} \ge 0$$

Then $\min\limits_{x\in[0,1]} \varphi = 1$ $\max\limits_{x\in[0,1]} \varphi = \dfrac{2^p}{2} = 2^{p-1}$. Then we can prove the inequality.

On the other hand, for $0<p<1$.

$$\varphi'(x) \leq 0$$

$$\Rightarrow \min_{x\in[0,1]} \varphi(x) = \varphi(1) = 2^{1-p}, \quad \max_{x\in[0,1]} \varphi(x) = \varphi(0) = 1$$