3. If f has a bounded derivative on $[a,b]$. show that

$V_a^b f \leq \|f'\|_\infty (b-a)$

pf: for any Partition $P = \{a = x_0 < x_1 \dots < x_n = b\}$,

$$V(f,P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{n} \|f'\|_\infty (x_i - x_{i-1}) = \|f'\|_\infty (b-a)$$

5. Complete the proof of Lemma 13.3

(i). $V_a^b f = 0 \iff V(f,P) = 0$ for any partition P

if f const, $V_a^b f = 0$ by definition

if $V_a^b f = 0$. $\underline{\text{for any fixed } x}$ Choose $\{a, x, b\}$ as the partition, then

$V(f,P) = |f(x) - f(a)| + |f(b) - f(x)| = 0$

Hereafter, $f(x) = f(a) = f(b)$ for any x

(ii) for any Partition P. we have. Then $V(cf,P) = |c| \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| = |c| V(f,P)$

$V(f,P) = \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|$

Therefore. $V_a^b(cf) = \sup_P V(cf,P) = |c| \sup_P V(f,P) = |c| V_a^b f$

(iv) for any partition P, $V(fg,P) = \sum_{i=1}^{n} |f(t_i) g(t_i) - f(t_{i-1}) g(t_{i-1})|$

$$\leq \sum_{i=1}^{n} |f(t_i)||g(t_i) - g(t_{i-1})| + |f(t_i) - f(t_{i-1})||g(t_{i})|$$

$$\leq \|f\|_\infty \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})| + \|g\|_\infty \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|$$

$$= \|f\|_\infty V(g,P) + \|g\|_\infty V(f,P)$$

Hereafter, $V_a^b(fg) \leq \|f\|_\infty V_a^b g + \|g\|_\infty V_a^b f$

(v) for any partition P. $V(|f|,P) = \sum_{i=1}^{n} \big||f(t_i)| - |f(t_{i-1})|\big|$

$$\leq \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| = V(f,P)$$

Therefore $V_a^b(|f|) \leq V_a^b f$

7. Suppose that $f \in B[a,b]$, if $V_{a+\varepsilon}^{b} f \le m$ for all $\varepsilon > 0$ does it follow that f is of bounded variation on $[a,b]$? Is $V_a^b f \le m$? If not, what additional hypotheses on f would make this so?

pf: Since f is bounded, we assume $\|f\|_\infty \le M_1$

for any partion $P = \{a = t_0 < t_1 < \dots < t_n = b\}$.

Choose $\varepsilon > 0$ such that $a < a+\varepsilon < t_1$. Consider $Q = \{a = t_0 < a+\varepsilon < t_1 < \dots < t_n = b\}$. then

$$v(f, P) \le v(f, Q) = |f(a+\varepsilon) - f(a)| + V_{a+\varepsilon}^{b} f \le V_{a+\varepsilon}^{b} f + 2M_1 \le M + 2M_1$$

Therefore. $V_a^b f \le M + 2M_1$

However, we can not get the conclusion $V_a^b f \le M$

Here is a counterexample: $a=0$, $b=1$

$$f(0) = \begin{cases} 1 & x=0 \\ 0 & x \in (0,1] \end{cases}$$

Then $V_{a+\varepsilon}^{b} f = 0$, and $V_a^b f = 1$.

If f is right continuous at a, i.e. $\lim_{\varepsilon \to 0^+} f(a+\varepsilon) = f(a)$. Then.

$v(f, P) \le |f(a+\varepsilon) - f(a)| + M$ Let $\varepsilon \to 0$, Then $v(f, P) \le M$

Hereafter, $V_a^b f \le M$

11. If $f_n \to f$ pointwise on $[a,b]$, show that $v(f_n, P) \to v(f, P)$ for any partition P of $[a,b]$. In particular, if we also have $V_a^b f_n \le K$ for all n, then $V_a^b f \le K$.

pf: for any partition P,

$$v(f, P) = \sum_{i=1}^{m} |f(t_i) - f(t_{i-1})|$$

$$= \sum_{i=1}^{m} \lim_{n\to\infty} |f_n(t_i) - f_n(t_{i-1})|$$

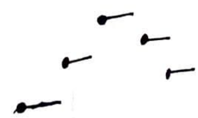
$$= \lim_{n\to\infty} \sum_{i=1}^{m} |f_n(t_i) - f_n(t_{i-1})| = \lim_{n\to\infty} v(f_n, P) \le K$$

Therefore, $V_a^b f \le K$

14. Let $I(x)=0$ if $x<0$ and $I(x)=1$ if $x\geq 0$, Given a sequence of scalars (c_n) with $\sum_{n=1}^{\infty}|c_n|<\infty$ and a sequence of distinct points (x_n) in $(a,b]$, define $f(x)=\sum_{n=1}^{\infty} c_n I(x-x_n)$ for $x\in[a,b]$. Show that $f\in BV[a,b]$ and that $V_a^b f=\sum_{n=1}^{\infty}|c_n|$

Pf : First of all, for any partition $P=\{a=t_0<\dots<t_m=b\}$

$$V(f,P)=\sum_{i=1}^{m}\left|\left[\sum_{n=1}^{\infty}c_n I(t_i-x_n)-\sum_{n\geq 1}^{\infty}c_n I(t_{i-1}-x_n)\right]\right|$$



for each $(t_{i-1},t_i]$, we have

$$c_n I(t_i-x_n)-c_n I(t_{i-1}-x_n)=\begin{cases} c_n & x_n\in(t_{i-1},t_i] \\ 0 & x_n\notin(t_{i-1},t_i]\end{cases}$$

Furthermore, for each x_n, there must exist a unique $1\leq i\leq m$ such that $x_n\in(t_{i-1},t_i]$, then $V(f,P)\leq\sum_{n\geq 1}^{\infty}|c_n|<\infty$, therefore, $V_a^b f\leq\sum_{n=1}^{\infty}|c_n|$

Now, we prove the inverse direction:

for any fixed N, reorder the sequence $\{x_1,\dots,x_N\}$, we still use $\{x_1,\dots,x_N\}$ to denote i.e.

Defn the Partition $P_N=\{a=x_0<x_1\dots<x_N<b\}$, then

$$V(f,P_N)=\sum_{i=1}^{N}\left|\sum_{n=1}^{\infty}c_n I(x_i-x_n)-\sum_{n=1}^{\infty}c_n I(x_{i-1}-x_n)\right|+\left|\left[\sum_{n=1}^{\infty}c_n I(b-x_n)-\sum_{n=1}^{\infty}c_n I(x_N-x_n)\right]\right|$$

$$\geq\sum_{i=1}^{N}\left|\left[\sum_{n=1}^{N}c_n I(x_i-x_n)-\sum_{n=1}^{N}c_n I(x_{i-1}-x_n)\right]\right|-\left|\sum_{n=N+1}^{\infty}c_n I(b-x_n)-\sum_{n=N+1}^{\infty}c_n I(x_N-x_n)\right|$$

$$-\sum_{i=1}^{N}\left|\sum_{n=N+1}^{\infty}c_n I(x_i-x_n)-\sum_{n=N+1}^{\infty}c_n I(x_{i-1}-x_n)\right|$$

$$\geq\sum_{n=1}^{N}|c_i|-\left|\sum_{n\geq N+1}^{\infty}c_n I(b-x_n)-\sum_{n\geq N+1}c_n I(x_N-x_n)\right|-\sum_{i=1}^{N}\left|\sum_{n=N+1}^{\infty}c_n I(x_i-x_n)-\sum_{n=N+1}^{\infty}c_n I(x_{i-1}-x_n)\right|$$

Since for each $n\geq N+1$, x_n must belong to a unique interval set in the Partition P_N,

we have

$$V(f,P_N)\geq\sum_{n=1}^{N}|c_i|-\sum_{n=N+1}^{\infty}|c_i|=\sum_{n=1}^{\infty}|c_i|-2\sum_{n=N+1}^{\infty}|c_i|$$

Let $N\to+\infty$, we have $\sum_{n=N+1}^{\infty}|c_i|\to 0$, Therefore, $V_a^b f\geq\lim_{N\to+\infty}V(f,P_N)=\sum_{n=1}^{\infty}|c_i|$

16. Given $f \in BV[a,b]$, define $g(x) = f(x+)$ for $a \le x < b$ and $g(b) = f(b)$. Prove that g is right continuous and of bounded variation on $[a,b]$.

(Notice $f \in BV[a,b] \Rightarrow f(x+)$ exists)

Pf: Our goal is to prove:

$$g(x) = \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^+} \lim_{t \to x^+} f(t) = \lim_{x \to x_0^+} g(x)$$

(That's what we want to prove)

Given any $\varepsilon > 0$ and $\forall x_0 \in [a, b)$, chose $\delta > 0$ s.t. $|f(y) - f(x_0+)| < \varepsilon/2$ for all $\{|y - x| < \delta \cap y > x_0\}$. For any $y \in (x, x+\delta)$, Let $\{x_n\} \to x_0^+$
$\{y_n\} \to y^+$

Then there exists an integer N such that $x_n, y_n \in (x, x+\delta)$ for all $n > N$. Then

$$|f(x_n) - f(y_n)| \le |f(x_n) - f(x+)| + |f(y_n) - f(x+)| < \varepsilon.$$

We have

$$|g(x) - g(y)| = |\lim_{n\to\infty} f(x_n) - \lim_{n\to\infty} f(y_n)| \le \lim_{n\to\infty} |f(x_n) - f(y_n)| \le \varepsilon \quad \text{for } y \in (x, x+\varepsilon)$$

Hence, g is right continuous. Notice that. we didn't use $f \in BV[a,b]$ here.

Moreover, we are ready to prove $g \in BV[a,b]$.

The first method:

for any partion P such that $a = x_0 < \dots < x_n = b$. Then.

$$V(g, P) = \sum_{i=1}^{n} |f(x_i+) - f(x_{i-1}+)| \quad \text{Here. we denote } f(b+) = f(b)$$

for each x_i with $0 \le i \le n-1$, choose $\tilde{x_i}$ such that $x_i < \tilde{x_i} < x_{i+1}$. and $|f(\tilde{x_i}) - f(x_i+)| < \frac{\varepsilon}{n}$.

Let $Q = \{a\} \cup \{\tilde{x_i}\} \cup \{b\}$

Then $V(g, P) \le 2\frac{\varepsilon}{n} \cdot n + V(f, Q)$.

Hence, since f bdd variation, we have. g is of bdded variation

The second method:

By Jordan's theorem, $f = p(x) - n(x)$. p and n are increasing.

Then $f(x+) = p(x+) - n(x+)$ and $p(x+)$, $n(x+)$ also increasing

$\Rightarrow g(x*) = f(x+)$ is bdded variation.

19. Suppose that f has a continuous derivative on $[a,b]$

(a) Use the mean value theorem to show that $V(f, P)$ can be written as a Riemann sum for $|f'|$ over P

(b) Show that $V_a^b f = \int_a^b |f'(t)|\,dt$

(c) Conclude that $p(x) = \int_a^b \{f'\}^+ dt$ and $n(x) = \int_a^b \{f'\}^-(t)\,dt$

Pf: (a) for any partition P such that $a = t_0 < \dots < t_n = b$. by Mean Value Theorem

$V(f, P) = \sum_{i=1}^n |f(t_i) - f(t_{i-1})| \underset{MVT}{=} \sum_{i=1}^n |f'(s_i)|(t_i - t_{i-1})$. Here $s_i \in [t_{i-1}, t_i]$

(b) for any partition P

$V(f, P) = \sum_{i=1}^n |f(t_i) - f(t_{i-1})| = \sum_{i=1}^n \left| \int_{t_{i-1}}^{t_i} f'(t)\,dt \right| \le \sum_{i=1}^n \int_{t_{i-1}}^{t_i} |f'(t)|\,dt$

$= \int_a^b |f'(t)|\,dt$.

Choose partition $Q = \{a = s_0 < \dots < s_N = b\}$ such that $f(x)$ is monotone over each interval $[s_{i-1}, s_i]$, $1 \le i \le N$.

Then $V(f, Q) = \sum_{i=1}^N |f(s_i) - f(s_{i-1})| = \sum_{i=1}^N \left| \int_{s_{i-1}}^{s_i} f'(t)\,dt \right| = \sum_{i=1}^N \int_{s_{i-1}}^{s_i} |f'(t)|\,dt$

$= \int_a^b |f'(t)|\,dt$

Hence, $V_a^b f = \int_a^b |f'(t)|\,dt$

(c) $p(x) = \frac{1}{2}(V(x) + f(x) - f(a)) = \frac{1}{2}\left(\int_a^x |f'(t)|\,dt + \int_a^x f'(t)\,dt\right) = \int_a^b \{f'\}^+ dt$

$n(x) = \frac{1}{2}(V(x) - f(x) + f(a)) = \frac{1}{2}\left(\int_a^x |f'(t)|\,dt - \int_a^x f'(t)\,dt\right) = \int_a^b \{f'\}^-(t)\,dt$