HW 2 Chap 14: 3, 6, 7, 9, 11, 25, 26

$\boxed{3.}$ for $f \in R_\alpha[a,b]$, for any partition P, we have

$$U(|f|, P) - L(|f|, P) \le U(f, P) - L(f, P)$$

Since $w(|f| : (x_{i-1}, x_i)) \le w(f : (x_{i-1}, x_i))$

Hence by Thm 14.4 (Riemann's Condition), we have.

$$|f| \in R_\alpha[a,b].$$

Moreover, for $\forall$ partition P.

$$|U(f, P)| \le U(|f|, P)$$

Hence, $\left|\int_a^b f\, d\alpha\right| \le \int_a^b |f|\, d\alpha$.

$\boxed{6}$ (a) $\Rightarrow$ if $f \in R_\alpha[-1,1]$, then for $\forall \frac{1}{n}$, $\exists$ partition P_n s.t

$$U(f, P) - L(f, P) < \frac{1}{n} \quad \text{for all } P_n \subset P$$

choose. $\tilde{P}_n = \{P_n\} \cup \{0\}$, Let $S_n = \min\{\tilde{P}_n \cap \{x > 0\}\}$

Then $L(f, \tilde{P}_n) = \inf\limits_{[0, S_n]} f$, $U(f, \tilde{P}_n) = \sup\limits_{[0, S_n]} f$

$$\Rightarrow \sup_{[0,S_n]} f - \inf_{[0,S_n]} f \le \frac{1}{n}. \quad \text{for } \forall n$$

$$\Rightarrow f(0+) = f(0).$$

$\Leftarrow$ if. $f(0+) = f(0)$, then. $\forall \varepsilon > 0$. $\exists \delta > 0$. if $0 < x < y < \delta$, then

$$|f(x) - f(y)| < \varepsilon.$$

Choose partition P such that $P = \{-1, 0, \delta, 1\}$

Then. $U(f, P) - L(f, P) \le \varepsilon$.

Then $f \in R_\alpha[-1,1]$

(b), (c) Same as the part (a)

(d) If $f \in R_r[-1,1]$, by (c), f cts at 0, then $f \in R_\alpha$, and $f \in R_\beta$

Moreover,

$$\int_{-1}^{1} f d\alpha = \lim_{s \to 0^+} f(s_0) = f(0).$$

$$\int_{-1}^{1} f d\beta = \lim_{s \to 0^-} f(s_0) = f(0)$$

$$\int_{-1}^{1} f d\gamma = \frac{1}{2}\left(\lim_{s \to 0^-} f(s) + \lim_{s \to 0^+} f(s)\right) = f(0)$$

Hence,

$$\int_{-1}^{1} f d\alpha = \int_{-1}^{1} f d\beta = \int_{-1}^{1} f d\gamma = f(0)$$

$\boxed{7}$ pf: Since f cts at each x_i, then for $\forall \varepsilon > 0$, $\exists \delta_i$ s.t

if $|x - x_i| \le \delta_i$ and $|y - x_i| \le \delta_i$, then

$$|f(x) - f(y)| \le \varepsilon.$$

Construct $P = \{x_0, x_1, \dots, x_n\} \cup \{x_0 + \delta_0, x_1 - \delta_1, x_1 + \delta_1, x_2 - \delta_2, \dots, x_{n-1} + \delta_{n-1}, x_n - \delta_n\}$

Then, $\quad U(f, Q) - L(f, Q) \le \varepsilon \sum_{i=0}^{n} d_n < K\varepsilon$

$\Rightarrow f \in R_\alpha.$

Moreover.

$$\inf_{[x_i - \delta_i, x_i + \delta_i]} f(x) \le f(x_i) \le \sup_{[x_i - \delta_i, x_i + \delta_i]} f(x)$$

$$\Rightarrow L(f, P) \le \sum_{i=0}^{n} f(x_i) d_i \le U(f, P)$$

$$\Rightarrow \int_a^b f d\alpha = \sum_{i=0}^{n} f(x_i) d_i$$

[9] pf: Since α cts on $[a,b]$, $\Rightarrow$ α uniformly cts.

For $\forall \varepsilon>0$. construct partion P s.t the length of each interval $<\delta$.

and $\Delta\alpha_i<\varepsilon$. , Since f is monotone, then

$$U(f,P)-L(f,P)=\sum_{i=1}^{n}|f(t_i)-f(t_{i-1})|\Delta\alpha_i<\varepsilon|f(b)-f(a)|$$

$\Rightarrow f\in R_\alpha[a,b]$

[11] pf: Since $f\in R_\alpha[a,b]$. then

$m[\alpha(b)-\alpha(a)]\le L(f,P)\le U(f,P)\le M[\alpha(b)-\alpha(a)]$, then.

$$m[\alpha(b)-\alpha(a)]<\int_a^b f\,d\alpha<M[\alpha(b)-\alpha(a)]$$

$$\Rightarrow m<\overset{C}{\overbrace{\frac{\int_a^b f\,d\alpha}{[\alpha(b)-\alpha(a)]}}}<M$$

Moreover, if f is cts. on $[a,b]$. by Extreme Value Theorem $\inf f=m'$. $\sup f=M'$, and $m\le m'\le M'\le M$. $\Rightarrow m'\le C$

By Intermediate Value Theorem. $\exists\, x_0\in[a,b]$ s.t $f(x_0)=C$

[25] Here. we can use the. Homework 6 (problem 6)

Construct $\alpha=\chi_{[0,1]}$ on $[-1,1]$; chose $f(x)=\begin{cases}0, & x\in[-1,0]\\ x & x\in[0,1]\end{cases}$

then $\int_{-1}^{1}|f|\,d\alpha=f(0+)=f(0)=0.$

Even if α cts.

$f=\begin{cases}0 & x\in[-1,0]\\ x & x\in[0,1].\end{cases}$ $\quad\alpha=\begin{cases}x & x\in[-1,0]\\ 0, & x\in[0,1]\end{cases}$

Here. $\int_{-1}^{1}|f|\,d\alpha.=0$

26 Pf: if $f(x_0) \neq 0$. then assume $|f(x_0)| \geq \varepsilon$.

Then $\exists\ \delta > 0$. s.t if $|x - x_0| < \delta$. $|f(x) - f(x_0)| \leq \frac{\varepsilon}{2}$,

And $|f(x_0)| \geq \frac{\varepsilon}{2}$.

$\Rightarrow \int_a^b |f(x)| dx \geq \int_{x_0-\delta}^{x_0+\delta} |f(x)| dx > \varepsilon \cdot \delta$.

(if $x_0 = a$ or b. then we choose the one-side interval)

① Hence. if $\int_a^b |f(x)| dx = 0$. and $f \in C[a,b]$ $\Rightarrow f \equiv 0$ on $[a,b]$

② $f.\ g \in C[a,b]$ and $\lambda \in \mathbb{R}$,

$\int_a^b |f+g| dx \leq \int_a^b |f| dx + \int_a^b |g| dx$

③ $\int_a^b |\lambda f(x)| dx = |\lambda| \int_a^b |f(x)| dx$

$\Rightarrow \|f\| = \int_a^b |f(x)| dx$ defines a norm on $C[a,b]$.

But it doesn't define a norm on $\mathcal{R}[a,b]$.

choose $f(x) = \begin{cases} 0 & \text{on } [a,b) \\ 1 & x = b \end{cases}$

we have $\int_a^b |f| dx = 0$. but $f \neq 0$