hw7 : chap 17 35, 41, 44, 45

[35] $\forall n \geq 1$. Define

$f_n : [0,\infty) \to \{0,1\}$, $f_n(x) = \chi_{[n-1,n]}$

$f_n \to 0$ pointwise.

Suppose $\exists\ F \subseteq \mathbb{R}$: $f_n \xrightarrow{\text{uniformly}} 0$ on F i.e.

$\forall \varepsilon > 0 .\ \exists N,\ \forall n \geq N,\ \forall x \in F,\ |f_n(x)| < \varepsilon$.

for $\varepsilon = 1$, $\exists N$, $\forall n \geq N$, $\forall x \in F$, $|f_n(x)| < 1 \Rightarrow x \notin [N,\infty)$

$\Rightarrow F \subseteq [0,1]$. and then $m(\mathbb{R} \setminus F) \geq m([N,\infty)) = \infty$.

[41] pf: $\mathbb{R} \setminus E$ is open, it can be written as a union of pairwise disjoint open interval

$((-\infty, a)$ or (a,∞) or $(a,b))$

① (a,b) : construct $g(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$, $x \in (a,b)$

② (a,∞) : $g(x) = g(a)$

③ $(-\infty, a)$: $g(x) = g(a)$

We can see. $\sup_{x \in \mathbb{R}} |g(x)| \leq \sup_{x \in E} |f(x)|$.

[44] For each n, choose g_n s.t $m\{|f-g_n| \geq 2^{-n}\} < 2^{-n}$.

Let $E_n = \{|f-g_n| \geq 2^{-n}\}$, $E = \limsup_{n\to\infty} E_n = \bigcap_{n\geq 1} \bigcup_{j\geq n} E_j$

if $x \notin E$, then $\exists n \geq 1$ s.t for $\forall j \geq n$, $x \in E_j^c$

$\Rightarrow |f - g_j(x)| \leq 2^{-j}$ for $j \geq n$.

$\Rightarrow \lim_{j\to+\infty} g_j(x) = f(x)$ for $x \notin E$

$\Rightarrow g_n \to f$ except the set E.

Furthermore $m(E) \leq m(\bigcup_{j\geq n} E_j)$ for $\forall n$.

$$\leq \sum_{j\geq n} 2^{-j} \leq 2^{1-n}$$

Let $n \to \infty$

$\Rightarrow m(E) = 0$.

Therefore, we find a sequence of continuous function (g_n) on $[a,$

s.t $g_n \to f$ a.e on $[a,b]$

[45] pf: ① By Problem 44. $\exists g_n \to f$ a.e. on $[a,b]$
cts

② Egorov: $\exists E \subset [a,b]$ s.t $m(E) < \frac{\varepsilon}{2}$ s.t

$g_n \rightrightarrows f$ on $[a,b] \setminus E$

Actually, we can choose $E \subset O$ (open) s.t $m(O) < \varepsilon$.

Then $g_n \rightrightarrows f$ on $[a,b]\setminus O$. $\Rightarrow f$ cts on $[a,b]\setminus O$

③ Since $[a,b]\setminus O$ is compact, By Problem 41, we can extend f to a continuous function g on $[a,b]$. Moreover,

$\{f \neq g\} \subseteq O \Rightarrow m\{f\neq g\} < \varepsilon$