HW 8 chap 18 : 3, 6, 9, 10, 11, 18, 22

$\boxed{3.}$ Define $\phi_n = \sum_{k=1}^{n} \frac{1}{k+1} \chi_{[k,k+1)}$. then $0 \le \phi_n \le \frac{1}{x}$ on $[1,\infty)$

Since, $\int_0^1 \phi_n = \sum_{k=1}^{n} \int_k^{k+1} \frac{1}{k+1} dx = \sum_{k=1}^{n} \frac{1}{k+1}$, we have

$\lim_{n\to\infty} \int_0^1 \phi_n = +\infty$

Furthermore. $\int_0^1 \frac{1}{x} = \sup \{ \int_1^\infty \varphi : 0 \le \varphi(x) \le \frac{1}{x} \}$

$\Rightarrow \int_0^1 \frac{1}{x} = +\infty$

$\boxed{6}$ $f_k - f_n$ increasing function. and converges pointwise to $f_1 - f$

$\int f_k - f_n \to \int f_k - f$ (By MCT)

$\lim_{n\to\infty} \int (f_k - f_n) = \int f_k - \int f$ (Here $0 \le f \le f_k$. f_k integrable $\Rightarrow f$ integrable)

$\Rightarrow \int f_k - \lim_{n\to\infty} \int f_n = \int f_k - \int f$

Since $\int f_k < \infty \Rightarrow \lim_{n\to\infty} \int f_n = \int f$

Example: $(f_n) = \sum_{k=0}^{n} \chi_{(k, k+\frac{1}{n})}$ f_n decreasing to 0

$\int f_n = \frac{\infty}{n} = \infty$ $\lim_{n\to\infty} \int f_n = \infty \neq 0 = \int \lim_{n\to\infty} f_n$.

$\boxed{9}$: $\int_E f = 0 \Rightarrow \int_{\mathbb{R}} f\chi_E = 0$ since $f\chi_E \geq 0$

$\Rightarrow f\chi_E = 0$ a.e $\Rightarrow f\chi_E = 0$ on $\mathbb{R}\setminus A$ with $m(A) = 0$

Since $f > 0$ a.e $\Rightarrow$ $f > 0$ on $\mathbb{R}\setminus B$ with $m(B) = 0$.

$\Rightarrow \chi_E = 0$ on $\mathbb{R}\setminus(A\cup B)$ $\Rightarrow E \subset A \cup B \Rightarrow m(E) \le m(A) + m(B) = 0$

$\boxed{10}$ ① $f_n = \chi_{[-n,n]} f$

Since $f \geq 0$, $f_n \uparrow f$, By Monotone convergence theorem, we have

$$\int f = \lim_{n\to\infty} \int f_n = \lim_{n\to\infty} \int_{-n}^{n} f \, dx$$

② Let $E_n = \{f \geq \frac{1}{n}\}$ Since $f \geq 0$, $E_n \subset E_{n+1}$

$\bigcup_{n=1}^{\infty} E_n = \{f > 0\}$, $f\chi_{E_n} \uparrow f\chi_{\{f>0\}}$ (since if $f=0$ $f\chi_{E_n} = 0 \Rightarrow f=0$

By MCT, we have

$$\int_{\{f>0\}} f = \lim_{n\to\infty} \int_{\{f \geq \frac{1}{n}\}} f$$

$\boxed{11}$ $f \geq 0$. Let $E_n = \{f \leq n\}$, then $\bigcup_{n=0}^{\infty} E_n = \mathbb{R}$ and $E_n \subset E_{n+1}$.

$f\chi_{E_n} \uparrow f$. By MCT, $\lim_{n\to\infty} \int f\chi_{E_n} = f$,

i.e $\int_{\mathbb{R}} f = \lim_{n\to\infty} \int_{\{f\leq n\}} f$.

$\boxed{18}$ Let $f_n = \chi_{[n,n+1)}$ then $\lim_{n\to\infty} f_n = 0$

$\Rightarrow \int \liminf_{n\to\infty} f_n = 0$

Meanwhile, $\int f_n = 1 \Rightarrow \liminf_{n\to\infty} \int f_n = 1$.

Then $\liminf_{n\to\infty} \int f_n > \int \liminf_{n\to\infty} f_n$

22. ① By Fatou's Lemma.

$$\int f \chi_E \leq \liminf_{n\to\infty} \int f_n \chi_E$$

② By Fatou's Lemma:

$$\int f \chi_{E^c} \leq \liminf_{n\to\infty} \int f_n \chi_{E^c} \leq \lim_{n\to\infty} \sup \int f_n - \limsup_{n\to\infty} \int_E f_n$$

$$\leq \int f$$

① + ② $\quad \int f \leq \int f + \liminf_{n\to\infty} \int_E f_n - \limsup_{n\to\infty} \int_E f_n$

Since $\int f = \lim_{n\to\infty} \int f_n < \infty$

$\Rightarrow \int f \chi_E = \liminf_{n\to\infty} \int f_n \chi_E = \limsup_{n\to\infty} \int f_n \chi_E = \lim_{n\to\infty} \int_E f_n.$

Example : $f_n(x) = n^2 \chi_{(0,\frac{1}{n}]}(x) + 1, \quad f_n \to 1$ on $(0, \infty)$

$$\lim_{n\to\infty} \int_0^\infty f_n = \int_0^\infty f = +\infty$$

But $\lim_{n\to\infty} \int_0^1 f_n = \lim_{n\to\infty} \left(\int_0^{\frac{1}{n}} (n^2+1) + \int_{\frac{1}{n}}^1 1 \right) = 1 + n \neq \int_0^1 f = 1$