We have set the stage for the Lebesgue integrals in the previous two chapters; now it is time for the star to make her entrance .

Quick review :

From Riemann integral → to Lebesgue integral.

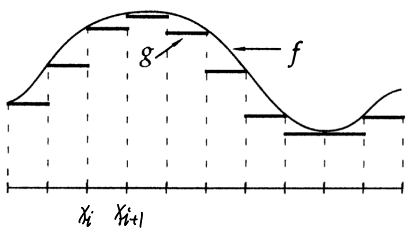
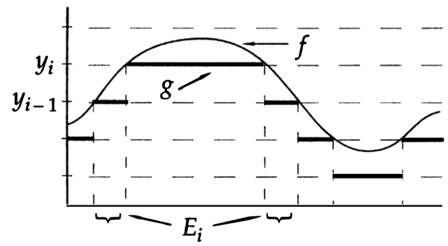


Figure 16.1



$\int_a^b f(x)\,dx \approx \sum f(\tilde{x}_i)\,(x_{i+1}-x_i)$

f "almost" continuous ↓

$\int_a^b f \approx \sum y_i \, m(y_i \le f < y_{i+1})$

measurable sets. ↓

Chp 17: measurable functions ← Chp 16: Lebesgue measure

We want the Lebesgue integral to satisfy :

(i) Based on measure : $\int \chi_E = m(E)$,

(ii) linear : $\int (\alpha f + \beta g) = \alpha \int f + \beta \int g$

(iii) monotone : $f \ge 0 \Rightarrow \int f \ge 0$ (or $f \ge g \Rightarrow \int f \ge \int g$)

(iv) be defined for a large class of functions.

(Riemann integrable fns at least)

For (iv):

→ Simple fns φ_n

↓ $0 \le \varphi_n \le f$, $\varphi_n \uparrow$

→ $\int f = \lim_{n\to\infty} \int \varphi_n$.

General : $f = f^+ - f^-$

$f_n \to f$ a.e.

Additional ⟹ $\int f = \lim_n \int f_n$

Chp 18 Outline

1) Simple fns

2) Nonnegative fns (Monotone Convergence)

3) General fns (Dominated Convergence)

17. Simple functions. S

$$\varphi = \sum_{i=0}^{n} a_i \chi_{E_i} \qquad \{a_i\} \subseteq \mathbb{R}, \quad E_i \text{ measurable}$$

<u>Standard representation</u> (unique) $\varphi = \sum_{i=0}^{n} a_i \chi_{A_i}$ $\quad \{a_i\}$ distinct, A_i disjoint. $(A_i = \{\varphi = a_i\})$

<u>Def</u>: A simple fn φ is <u>Lebesgue integrable</u> if the set $\{\varphi \neq 0\}$ has finite measure.

Define its LI as: $\int \varphi = \sum_{i=0}^{n} a_i \, m(A_i) = \sum_{i=0}^{n} a_i \, m\{\varphi = a_i\}$.

(convention: $0 \cdot \infty = 0$: $a_0 = 0$, $a_0 \cdot m(A_0) = 0$).

Denote $\varphi \in L^1 \cap S$

<u>Example</u>: $\int \chi_{\mathbb{Q}} = 0 \cdot m(\mathbb{Q}^c) + 1 \cdot m(\mathbb{Q}) = 0$.

- The definition does NOT depend on the standard (or any) presentation.

<u>Lemma 18.1</u> Let φ be an integrable simple fn, and let $\varphi = \sum_{i=1}^{n} b_i \chi_{E_i}$ be any representation w/ $\{E_i\}$ disjoint and measurable. Then, $\int \varphi = \sum_{i=1}^{n} b_i \, m(E_i)$.

<u>Pf</u>. Note: $\{\varphi = a\} = \bigcup_{\{i : b_i = a\}} E_i \quad \Rightarrow \quad a \, m\{\varphi = a\} = \sum_{\{i : b_i = a\}} b_i \, m(E_i)$.

This equality is true even if $m(E_i) = \infty$, which happens only when $b_i = a = 0$.

Consequently, $\int \varphi = \sum_{a \in \mathbb{R}} a \, m\{\varphi = a\} = \sum_{a \in \mathbb{R}} \sum_{\{i : b_i = a\}} b_i \, m(E_i) = \sum_{i=1}^{n} b_i \, m(E_i)$. #

(↓ finite sum)

- The integral is both linear and positive on integrable simple fns.

<u>Lemma 18.2</u> If φ & $\psi \in L^1 \cap S$ then for $\alpha, \beta \in \mathbb{R}$, we have $\int (\alpha\varphi + \beta\psi) = \alpha \int \varphi + \beta \int \psi$.

If $\varphi \geq \psi$ a.e., then $\int \varphi \geq \int \psi$.

<u>Pf</u>: "The heart of the matter here is to <u>find a common partition</u> for the representations of φ & ψ".

Let $\varphi = \sum_{i=0}^{n} a_i \chi_{A_i}$, $\psi = \sum_{j=0}^{k} b_j \chi_{B_j}$, where $a_0 = 0 = b_0$. $\{a_i\}_{i=1}^{n}$ distinct, $\{b_j\}_{j=1}^{k}$ distinct.

$A_i = \{\varphi = a_i\}$, $\quad B_j = \{\psi = b_j\}$ measurable.

Then, $\mathbb{R} = \bigcup_i A_i = \bigcup_j B_j$, both disjoint union, all but A_0 & B_0 have finite measure.

$= \bigcup_i \bigcup_j (A_i \cap B_j)$, disjoint union, all but $A_0 \cap B_0$ have - - - .

$$\varphi = \sum_{i=0}^{n} \sum_{j=0}^{k} a_i \chi_{A_i \cap B_j} \qquad \psi = \sum_i \sum_j b_j \chi_{A_i \cap B_j}$$

$$\Rightarrow \alpha\varphi + \beta\psi = \sum_i \sum_j (\alpha a_i + \beta b_j) \chi_{A_i \cap B_j}$$

$$\int(\alpha\varphi+\beta\psi) = \sum_i\sum_j (\alpha a_i+\beta b_j)\, m(A_i\cap B_j)$$
$$= \alpha\sum_i\sum_j a_i\cdot m(A_i\cap B_j) + \beta\sum_i\sum_j b_j\, m(A_i\cap B_j)$$
$$= \alpha\int\varphi + \beta\int\psi.$$

Finally, if $\varphi-\psi\geq 0$ a.e., $\varphi-\psi = \sum_i\sum_j (a_i-b_j)\,\chi_{A_i\cap B_j}$, $c_{ij}=a_i-b_j$, $m(A_i\cap B_j)=0$ if $c_{ij}<0$.

$$\Rightarrow \int\varphi-\int\psi = \int(\varphi-\psi) = \sum_{i,j:\, c_{ij}\geq 0} c_{ij}\, m(A_i\cap B_j)\geq 0. \qquad \#.$$

Cor 18.3 $\quad \int\sum_{i=1}^n a_i\chi_{E_i} = \sum_{i=1}^n a_i\, m(E_i)$ if $\{a_i\}\subseteq\mathbb{R}$, $m(E_i)<\infty$, $\forall i$.

Non-negative fns.

Def: If $f:\mathbb{R}\to[0,\infty]$ is measurable, we define its Lebesgue integral over $\mathbb{R}$ by

$$\int f = \sup\left\{\int\varphi : 0\leq\varphi\leq f,\ \varphi \text{ simple and integrable}\right\}.$$

If $\int f<\infty$, then f is Lebesgue integrable on $\mathbb{R}$. Denote $f\in L^1(\mathbb{R},\overline{\mathbb{R}^+})$

Note: • It is possible that $\int f=\infty$: e.g. $f\equiv 1$ or $f=\chi_E$ w/ $m(E)=\infty$.

(b.c. $\int\chi_E \geq \sup_n\int\chi_{E\cap[-n,n]} = \sup_n m(E\cap[-n,n]) = m(E)=\infty$).

This integral exists ($=\infty$) for non-integrable fns.

• $\int f\geq 0$ by definition. So $f\geq g \Rightarrow \int f\geq\int g$.

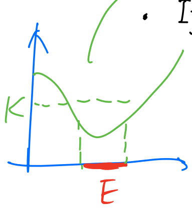
Define $\quad \int_E f = \int f\chi_E$, where E measurable, f nonnegative, measurable

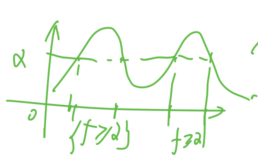
• If $m(E)=0$, then $\int_E f=0$ (b.c. $0\leq\varphi\leq f\chi_E$ (simple) $\Rightarrow\varphi=0$ a.e. $\Rightarrow\int\varphi=0$)

• If $f\geq 0$, $E\subset F$ measurable, then, $\int_E f\leq\int_F f$ (b.c. $f\chi_E\leq f\chi_F$)

• If $0\leq f\leq k$ on E, then $\int_E f\leq k\, m(E)$ (b.c. $f\chi_E\leq k\chi_E$).

• If $f\geq 0$ & measurable, then $\int f\geq\alpha\, m\{f\geq\alpha\}$, $\forall\alpha\geq 0$ (b.c. $f\geq\alpha\,\chi_{\{f\geq\alpha\}}$.)





$\Rightarrow$ Chebyshev's inequality 18.4.

$$\int|f|\geq\alpha\, m(|f|\geq\alpha),\ \forall\alpha\geq 0.$$

if $|f|$ is measurable.

Cor 18.5 If f is nonnegative & integrable, then f is finite a.e.

Pf. Note: $\{f=\infty\} = \bigcap_{n=1}^{\infty} \{f \geq n\}$,

$\{f \geq n\} \downarrow$ as $n \uparrow$. $\quad m\{f \geq n\} \leq \frac{1}{n}\int f \xrightarrow{n\to\infty} 0$ (since $\int f < \infty$).

Thus, $m(f=\infty) = \lim_{n\to\infty} m(f \geq n) = 0$. #.

Exe3: Prove that $\int_1^\infty \frac{1}{x}dx = \infty$ as a Lebesgue integral.

Pf: $= \sup\{\int \varphi : 0 \leq \varphi(x) \leq \frac{1}{x}, \forall x, \varphi \text{ simple \& integrable}\}$.

Let $\varphi_m(x) = \sum_{n=1}^{m} \frac{1}{n+1} \chi_{[n,n+1)}$.

Then $0 \leq \varphi_m(x) \leq \frac{1}{x}$, $\forall x \in [1,\infty)$

φ_m simple, $\int \varphi_m = \sum_{n=1}^{m} \frac{1}{n+1} < \infty$. $\Big\} \Rightarrow \int_1^\infty \frac{1}{x}dx \geq \sup_m \int \varphi_m = \sup_m \sum_{n=1}^{m} \frac{1}{n+1} = \infty$.

#.

Exe $\quad \int |f| = \int_0^\infty m(|f|>\alpha)\,d\alpha$

$\left[\int_0^\infty m(|f|>\alpha)\,d\alpha = \int_0^\infty \int 1_{\{|f|>\alpha\}}\,dm\,d\alpha \overset{\text{Fubini}}{=} \int\int_0^\infty 1_{\{|f|>\alpha\}}\,d\alpha\,dm = \int |f|\,dm. \right]$

$\int_0^{|f|} 1\,d\alpha = |f|$

- Monotone convergence & Fatou's Lemma. (Nonnegative Fn)
- General case
- Dominated Conv.
- Appr. of L^1

Motivations:

① Exchange order of limit & integral: $\lim \int = \int \lim$?

② measure ⟺ Integral.

$m \qquad f \geq 0 \qquad \mu(E) = \int_E f$

is μ a measure?

(nonnegative, monotone, $\mu(\phi)=0$, countably additive).

Lebesgue Measure: $\mu: \mathcal{M} \to \mathbb{R}^+$ — measurable sets $\{\phi, \mathbb{R}, E, E^c, \ldots\}$ (open subsets of σ-mean)

- $E \subseteq F \Rightarrow m(E) \leq m(F)$ (Monotone) ⇒ subadditive
- $m(E+x) = m(E)$ (Shift-invariant)
- countably additive $\{E_i\}$ pairwise disjoint

$m(\bigcup E_i) = \sum_i m(E_i)$

Monotone Convergence & Fatou's Lemma

(Integral commutes w/ increasing limits) → "in" ≤ "out"

Exe 4: Find $\{f_n\}$ s.t. $\int \lim_{n\nearrow\infty} f_n = 0$, but $\lim_n \int f_n = 1$. Ans. $f_n = 1_{[n,n+1]}$ or $f_n = \frac{1}{2}\sum_{|i|\le n} 1_{[i, i+\frac{1}{n}]}$

Lemma 18.6 Let φ be an integrable simple fn; $\{E_n\} \subset \mathcal{M}$ s.t. $E_n \subset E_{n+1}$.

Then, $\int_{\cup_{n=1}^\infty E_n} \varphi = \lim_{n\to\infty} \int_{E_n} \varphi$ i.e. $\mu(E) = \int_E \varphi$ is "continuous".

Rmk1: exe 8. $\mu: A \to [0,\infty]$: finite additive, set fn. on σ-algebra A. Then countably additive ⇔ continuous.

Rmk2: $\chi_{E_n} \nearrow \chi_E$. $\int \lim_n \varphi\chi_{E_n} = \lim_n \int \varphi \chi_{E_n}$

Pf: Write $\varphi = \sum_{i=1}^k a_i \chi_{A_i}$, $a_i \neq 0$, A_i disjoint. measurable. finite measure. Then,

$$\int_E \varphi = \int \varphi \chi_E = \int \sum_{i=1}^k a_i \chi_{A_i \cap E} = \sum_{i=1}^k a_i m(A_i \cap E)$$ continuity of m. $= \lim_{n\nearrow\infty} m(A_i \cap E_n)$

$$= \lim_{n\nearrow\infty} \sum_{i=1}^k a_i m(A_i \cap E_n) \quad \leftarrow \quad \sum_{i=1}^k \lim_n = \lim_n \sum_{i=1}^k$$

(the last sum is $\int_{E_n} \varphi$) #

MCT **Monotone Convergence Thm 18.7.** If $0 \le f_1 \le f_2 \le \cdots$, then $\int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n$.

Pf, Since $f_n \nearrow$, $f = \lim_n f_n$ exists, measurable & ≥ 0.

$\lim_{n\nearrow\infty} \int f_n$ exists and $\lim_{n\nearrow\infty} \int f_n \le \int f$ b.c. $\int f_n \le \int f_{n+1} \le \int f$, $\forall n$.

We only need to show that $\lim_n \int f_n \ge \int f \ge (1-\varepsilon)\int f$, $\forall \varepsilon > 0$

☆ It suffices to show $\lim_n \int f_n \ge (1-\varepsilon)\int \varphi$, $\forall\ 0 \le \varphi \le f$. φ simple. $\forall \varepsilon$.

(Getting to a much larger class, but easier to handle)

Let $E_n = \{f_n \ge (1-\varepsilon)\varphi\}$. Then, E_n measurable,

$E_n \subset E_{n+1}$ (b.c. $f_n \le f_{n+1}$)

$\bigcup_{n=1}^\infty E_n = \mathbb{R}$ (b.c. $f_n \to f > (1-\varepsilon)\varphi$).

Consequently: $\int f_n \ge \int_{E_n} f_n \ge \int_{E_n} (1-\varepsilon)\varphi = (1-\varepsilon)\int_{E_n}\varphi$

$n \nearrow \infty$: $\lim_{n\nearrow\infty} \int f_n \ \ge \ (1-\varepsilon)\int_{\mathbb{R}} \varphi$ by Lemma 18.6. #

Cor 18.8 If $f \geq 0$ meas. Then, $\exists \{\varphi_n\}$ simple fns s.t. $0 \leq \varphi_1 \leq \varphi_2 \cdots \leq f$, integrable

$$f = \lim_{n\to\infty} \varphi_n \quad \& \quad \int f = \lim_n \int \varphi_n.$$

Proof: $\{\psi_n\}$ from the Basic Construction; $\varphi_n = \psi_n 1_{[-n,n]}$. Then $\varphi_n \uparrow$, integrable, ✓✓✓. #

$$\left(\begin{array}{l} F_n = \{x \in D : f(x) > 2^n\} \\ E_{n,k} = \{x \in D : k 2^{-n} \leq f(x) < \frac{k+1}{2^n}\}, \quad k = 0, 1, \dots, 2^{2n}-1. \\ \varphi_n = 2^n \chi_{F_n} + \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \chi_{E_{n,k}} \end{array} \right).$$

$\Rightarrow$ f & $\int f$ are completely determined by the seq. $\{\varphi_n\}$. Thus, additivity of simple fns. extends to general f.

Cor 18.9 $f, g \geq 0$. meas. Then $\int f + g = \int f + \int g$

E & F disjoint, then, $\int_{E \cup F} f = \int_E f + \int_F f$;

$$\int f 1_{E\cup F} = \int f 1_E + f 1_F$$

Pf: Choose two seq. $\varphi_n \uparrow f$, $\psi_n \uparrow g$. Then, $\varphi_n + \psi_n \uparrow f+g$ and by MCT:

$$\int f+g = \lim \int \varphi_n + \psi_n = \lim \int \varphi_n + \lim \int \psi_n = \int f + \int g. \quad \#$$

Cor 18.11 (Beppo Levi Thm) $f_n \geq 0$, $\forall n$. meas. Then

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$$

Pf: $F_n \geq 0$, $\uparrow$. By MCT: $\int \lim F_n = \lim_n \int F_n$ #.

Back to measure view: $\mu(E) \doteq \int_E f$:
1. nonnegative ✓, monotone ✓: $\mu(E) \leq \mu(F)$ if $E \subset F$.
2. finite additive: Cor 18.9.
3. countably additive: $\mu(\bigcup E_n) = \sum_{n=1}^{\infty} \mu(E_n)$ if E_n disj.

Cor 18.12: $f \geq 0$ meas. Then $\mu(E) = \int_E f$ is a meas. on $\mathcal{M}$.

$$\int_{\bigcup E_n} f = \int \lim_n f \chi_{\bigcup_{k=1}^{n} E_k} = \lim_n \int f \chi_{\bigcup_k E_k}$$

Σg_k $\quad$ Σ

Lemma 18.10 $f \geq 0$ meas. Then $\int f = 0 \Leftrightarrow f = 0$ a.e. $\leftarrow$ ⊕ sets of measure 0.

Pf: ($\Rightarrow$): to show $m(f>0)$: $\{f>0\} = \bigcup_{n=1}^{\infty} \{f \geq \frac{1}{n}\}$, & $m(f \geq \frac{1}{n}) \leq n \int f = 0$. #

The limit may NOT always exist if NOT monotone. $\underline{\lim}_n = \lim_{n\to\infty} \inf_{k\ge n}$

<u>Fatou's Lemma</u> (18.13). $\{f_n\}$ meas. $f_n \ge 0$, $\forall n$. Then, $\int \liminf_n f_n \le \liminf_n \int f_n$.

<u>Proof</u>: Let $g_n = \inf_{k\ge n} f_k$. Then $g_n \ge 0$, meas. increasing; $\lim_n g_n = \underline{\lim} f_n$

$\xrightarrow{\text{MCT}}$ $\int \underline{\lim} f_n = \int \lim_n g_n = \lim_n \int g_n$.

$\int g_n \le \int f_k$, $\forall k \ge n$. $\Rightarrow \int g_n \le \inf_{k\ge n} \int f_k \Rightarrow \lim \int g_n \le \underline{\lim}_n \int f_n$ $\Big\}$ (*). #.

<u>Exe18</u>: show the strict inequality in Fatou's Lemma for $f_n = 1_{[n,n+1]}$.

<u>Exe19</u>: $f_n \ge 0$: $\overline{\lim}_n \int f_n \le \int \overline{\lim} f_n$ for $\{f_n\}$ uniformly bdd.

(Is it true after removing uniform bddness?)

Hw. Chp18 3, 6, 9, 10, 11, 18, 22.