The general case $f: \mathbb{R} \to \overline{\mathbb{R}}$ Lebesgue integral

Preview $\left\langle \begin{array}{l} \text{simple fn} \quad \varphi = \sum a_i 1_{A_i} \to \int \varphi = \sum a_i \cdot m(A_i) \\ f \geq 0 \quad \text{Def} \quad \int f = \sup_{\varphi \in S,\ 0 \leq \varphi \leq f} \int \varphi \end{array} \right.$

$\Delta 1$: general f?

$\Delta 2$: linear vector sp? $\int af + bg = a\int f + b\int g$.

$\Delta 3$: lattice? (closed under partial order: ~~$f \geq g \Rightarrow \int f \geq \int g$~~). $f \vee g \in L_1$, $f \wedge g \in L_1$

$\Delta 4$: Relation w/ Riemann Sum?

$\Delta 5$: sequence convergence?

$f \geq 0$ $\boxed{1}$

- $\int af = a\int f$. $a > 0$
- $\Delta 1$ $\int f + g = \int f + \int g$;
- $\int f \geq \int g$ if $f \geq g \geq 0$
- MCT $\int \lim_n f_n = \lim_n \int f_n$
- Fatou: $\int \underline{\lim}_n f_n \leq \underline{\lim}_n \int f_n$

- $f \in \mathcal{M}, f \geq 0$ $\int f = 0 \Rightarrow f = 0$ a.e.
- $\mu_f(E)$ is a meas. on $\mathcal{M}$, if $f \geq 0, f \in \mathcal{M}$

$$f = f^+ - f^- ; \quad f^+ = f \vee 0 , \quad f^- = f \wedge 0 ; \quad \text{They have disjoint support!}$$

$|f| = f^+ + f^-$

Def: f is Lebesgue integrable if both f^+ & f^- are L. integrable. (No $\infty - \infty$)

$$\int f = \int f^+ - \int f^- \qquad \int_E f = \int f \chi_E$$

$L_1 = L^1(\mathbb{R}) = \{ f : \mathbb{R} \to \overline{\mathbb{R}} \text{ integrable} \}$; $L_1 = L^1(E) = \{ f : E \to \overline{\mathbb{R}} \text{ integrable} \}$

Properties: f is integrable $\Leftrightarrow$ f^+, f^- are integrable.

$\Leftrightarrow \int |f| < \infty$, i.e. $|f|$ is integrable.

Rmk This is substantially different from Riemann integral e.g. $f = 2\, 1_{\mathbb{Q} \cap [0,1]} - 1 = \begin{cases} 1 & \mathbb{Q} \\ -1 & \mathbb{Q}^c_{\cap [0,1]} \end{cases} \notin R[0,1]$.

$|f| \equiv 1 \in R[0,1]$

Observations 18.14

(a) $\left| \int f \right| \leq \int |f|$ b.c. $\left. \begin{array}{l} \int f = \int f^+ - \int f^- \\ |a - b| \leq |a| + |b| \end{array} \right\} \Rightarrow \left| \int f \right| \leq \int f^+ + f^- = \int |f|$.

(b) If f is integrable ($\in L^1$), then $f < \infty$ a.e. i.e. $m\{|f| = \infty\} = 0$.

(c) If f is integrable ($\in L^1$) & $m(E) = 0$. Then $\int_E f = 0$. $\rightarrow$ we only need to consider $f < \infty$ a.e. when studying integrals.

(d) If f & g measurable, $|f| < |g|$ & g integrable, then f is integrable & $\int |f| \leq \int |g|$.

☆ (e): $f : [a,b] \to \mathbb{R}$ bdd, meas. Then, f is measurable. In particular, bdd Riemann integrable $\xrightarrow{\Delta}$ Lebesgue integrable meas:?

(f) f measur. Then, $\exists$ $g \in \mathcal{B}$ (Borel) s.t. $f = g$ a.e.. Thus, we only need $\mathcal{B}$ when studying integrals.

Proposition 18.15 · L_1 is a vector sp, a lattice;

· The Lebesgue integral is a positive, linear, real-valued function on the sp.

Proof: 1> Vector sp. Let $g, f \in L_1$, $a, b \in \mathbb{R}$. Then, $|af+bg| \le |a||f| + |b||g|$ a.e. (No ∞)

$\in L^1 \Leftarrow \in L^1$

2> Lattice: $f, g \in L_1$, $|f \vee g| \le |f| + |g|$ a.e.; similarly $f \wedge g$.

$\in L_1 \leftarrow \in L_1$

3> Linear.
$$\begin{cases} (af)^{\pm} = af^{\pm}, & \forall a > 0 \\ (af)^{\pm} = -af^{\mp}, & \forall a < 0. \end{cases} \Rightarrow \int af = \begin{cases} \int af^{+} + af^{-} & a > 0 \\ -\int af^{-} + a\int f^{+} & a < 0 \end{cases} = a\int f$$

$$\int f+g = \int (f^{+} - f^{-} + g^{+} - g^{-}) = \int f + \int g$$

$$\| \qquad \uparrow$$

$\int (f+g)^{+} - \int (f+g)^{-}$

To convert to additive properties of nonnegative fns

$(f+g)^{+} - (f+g)^{-} = f^{+} - f^{-} + g^{+} - g^{-}$ a.e.

$\Rightarrow (f+g)^{+} + f^{-} + g^{-} = (f+g)^{-} + f^{+} + g^{+}$ a.e.

$\int \cdots \quad \int \cdots$

positive: · $f \ge 0$, $f \in L_1$, then $\int f \ge 0$

· $f \ge g$, $f, g \in L_1$, then $f - g \ge 0 \Rightarrow \int f - g \ge 0 \Rightarrow \int f \ge \int g$. #

$\boxed{\text{Riemann} \hookrightarrow \text{Lebesgue}}$

Thm 18.16 $f \in R[a,b]$ bdd. $\Rightarrow f \in L_1[a,b]$ & $(R)\int_a^b f(x)dx = (L)\int_a^b f$

Connection: step fns $\varphi(x) = \sum_i c_i 1_{(t_i, t_{i+1})}$ $a = t_0 < t_1 \cdots < t_n = b$ $\varphi \in R[a,b] \cap L_1$ $(R)\int_a^b \varphi(x)dx = (L)\int_a^b \varphi$

Pf: Since $f \in R[a,b]$, $\exists \{l_n\} \{u_n\}$ step functions s.t. $l_n \le f \le u_n$, s.t.

$$\sup_n \int_a^b l_n = (R)\int_a^b f(x)dx = \inf_n \int_a^b u_n \quad (*)$$

1> f is meas: $l = \sup_n l_n \le f \le u = \inf_n u_n$

$$\sup_n \int_a^b l_n \le \int_a^b l \le \int_a^b u \le \inf_n \int_a^b u_n \Rightarrow \int (u - l) = 0 \xrightarrow{u - l \ge 0} u = l = f \text{ a.e.} \Rightarrow f \in \mathcal{M}.$$

$+ f$ bdd

$\Rightarrow f \in L^1$

2> $l_n \le f \le u_n \Rightarrow \sup_n \int_a^b l_n \le (L)\int_a^b f \le \inf_n \int_a^b u_n \longrightarrow (R)\int_a^b f(x)dx = (L)\int_a^b f$ #

· Lebesgue integral subsumes the Proper Riemann integral

NOT if improper.

Example 18.17 Improper Riemann integral $(R)\int_0^\infty \frac{\sin x}{x}\,dx$ exists.

Lebesgue integral $(L)\int_0^\infty \frac{\sin x}{x}\,dx$ DNE.

(Requires $\int_0^\infty \left|\frac{\sin x}{x}\right| dx < \infty$.)

Proof: $$(R)\int_0^\infty \frac{\sin x}{x}\,dx = \sum_{n=1}^\infty \int_{(n-1)\pi}^{n\pi} \frac{\sin x}{x}\,dx = \sum_{n=1}^\infty (-1)^{n-1} \int_{(n-1)\pi}^{n\pi} \frac{|\sin x|}{x}\,dx$$

$x = (n-1)\pi + z$

$|\sin((n-1)\pi + z)| = |\sin z|$

$$= \sum_{n=1}^\infty (-1)^{n-1} \underbrace{\int_0^\pi \frac{|\sin z|}{(n-1)\pi + z}\,dz}_{C_n}$$

$< \infty$ b.c. $C_n \le \frac{1}{(n-1)}$ in the alternating series.

$$(L)\int_0^\infty \left|\frac{\sin x}{x}\right| dx \overset{MCT}{=} \sum_{n=1}^\infty \int_{(n-1)\pi}^{n\pi} \frac{|\sin x|}{x}\,dx \ge \sum_{n=1}^\infty \frac{1}{n\pi} \underbrace{\int_0^\pi |\sin x|\,dx}_{>0} = \infty. \quad \#$$

$$\ge \frac{1}{n\pi}\int_0^\pi |\sin x|\,dx$$

Rmk: $f \in L^1$ requires $|f| \in L^1$, No alternating (conditional convergent → R; absolute convergent)

· No "improper" Lebesgue integral: Nothing about bdd function or the set assumption.

. L_1 is NOT an algebra b.c. $f(x) = x^{-\frac{1}{2}}$ $0<x<1$, $\in L_1[0,1]$, but f^2 is NOT.

$\boxed{\text{Norm}}$

Lemma 18.18. $f, g \in L_1$ Then $f = g$ a.e. $\overset{①}{\Leftrightarrow} \int |f-g| = 0 \overset{②}{\Leftrightarrow} \int_E f = \int_E g$, $\forall E \in \mathcal{M}$.

Pf: $|f-g| = 0$ a.e.

$\left|\int_E f - g\right| \le \int_E |f-g| \le \int |f-g| = 0$

③ ⇒ ②: $E = \{f - g > 0\}$

Then $\int |f-g| = \int_E f-g + \int_{E^c} g-f$

$= 0 + 0 = 0.$ #

Norm on L_1: $\|f\|_1 = \int |f|$.

Pf: (i) $0 \le \|f\|_1 < \infty$, $\forall f \in L_1$ ✓

(ii) $\|f\|_1 = 0$ iff $\underline{f = 0 \text{ a.e.}}$ $\longleftrightarrow$ L_1 equivalent classes

$[f] = \{g \in L_1 : f = g \text{ a.e.}\}$

(iii) $\|\alpha f\|_1 = |\alpha|\,\|f\|_1$

(iv): triangle inequality. $\|f+g\|_1 \le \|f\|_1 + \|g\|_1$

$\boxed{18.14\ (C)}$